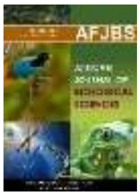
https://doi.org/10.48047/AFJBS.6.13.2024. 3269-3285

African Journal of Biological Sciences



APPLICATIONS OF PYTHAGOREAN FUZZY SUB ALGEBRAS OF BCK/BCI-ALGEBRA

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Article History

Volume 6, Issue 13, 2024

Received: 18June 2024

Accepted: 02July 2024

doi:*10.48047/AFJBS.6.13.2024.* 3269-3285

Abstract: In this paper, the concept of Pythagorean fuzzy set is investigated and compared with other types of uncertainty sets. We define Pythagorean fuzzy sub algebras in BCK-algebras and BCI-algebras and study their properties. A given Pythagorean fuzzy sub algebra is used to create a new Pythagorean fuzzy sub algebra. Also we obtain the intersection of two Pythagorean fuzzy sub algebras to be a Pythagorean fuzzy sub algebra is proved and an example is given to show that the union of two Pythagorean fuzzy sub algebras may not be a Pythagorean fuzzy sub algebra. The characterization of cut set is also used in Pythagorean fuzzy sub algebra. The homomorphic image and pre image of Pythagorean fuzzy sub algebra is discussed. It turns out that Pythagorean fuzzy sub algebra is a subclass of Fermatean fuzzy sub algebra.

2020 AMS classification: 03G25, 06F35, 08A72.

Keywords: fuzzy set, Pythagorean fuzzy set, BCK-algebra, BCI-algebra, union, intersection, sub algebra, cut set, pre image.

1.Introduction

An intuitionistic fuzzy set (IFS) is a set developed to handle problems related to imprecise and incomplete information [7]. This set was introduced by Atanassov, which is a generalization of the fuzzy set (FS) theory [30]. In FS, an element is marked by the presence of its membership (M) degree or value (i.e., the non-membership (N) degree is directly complemented to it). Mean while, in IFS, it is indicated by the presence of its M and N degrees, where the sum of the two can be less than one (i.e., any hesitancy or incomplete information is allowed). This makes IFS more flexible and covers more uncertain events in the decision-making process. Several studies have been conducted to expand the IFS, including in aggregation operators [26] and correlation coefficient [17], to mention a few. In addition, many authors have applied the IFS to decision-making problems [1,13]. IFS has experienced numerous developments, especially in terms of the relationship between M and N degrees. Initially, the IFS met the condition $M + N \leq 1$. However, to cater for the issue beyond this inequality (i.e., $M + N > 1$), Yager [28] then defined the Pythagorean fuzzy sets (PFS), which changed the constraining relation to $M^2 + N^2 \leq 1$. Prior to that, Atanassov[8] proposed IFS of second type to deal with the same issue. In 2011, Ciucci [15] introduced the term orthopair as an alternative pair of M and N degrees. This gives rise to the generalized orthopair fuzzy sets or called q-rung orthopair fuzzy sets (q-ROFS), which satisfy $Mq + Nq \leq 1$ for any q positive integers [29]. Vassilev et.al [25] defined a similar concept called IFS of q-type to generalize the IFS. Note that this set can be reduced to IFS for $q = 1$, PFS for $q = 2$ and Fermatean fuzzy sets (FFS), which is another special form of q-ROFS with $q = 3$ [24]. Similarly, several studies have explored the q-ROFS in the cases of aggregation operations [21,2], similarity measures [16,5] and some applications in decision-making problems [4,3]. In general, the expression of q-ROFS is acknowledged to provide greater flexibility and expressive power for decision-makers in representing their preferences compared to IFS [29]. In 1989, IFS was expanded from what was originally a singular point into an area in an intuitionistic fuzzy interpretation triangle (IFIT) with a rectangular shape called interval-valued IFS(IVIFS) [10]. The main motivation for this extension was to deal with imprecise of M and N values. Recently, Atanassov introduced another extension of M and N interpretation into a circle called circular IFS (CIFS) [9]. This set is characterized by a 3-tuple containing M, N and radius for each element. The difference with IFS lies in the existence of a circular imprecision area with radius r. Compared to IVIFS, CIFS has an equidistant centre point and boundary, which is not necessarily true for IVIFS, as their boundaries can take various shapes and distances from the centre point. The CIFS theory is still at an early stage of its development. Hence, not much research has been conducted on it. Initially, Atanassov [9] defined the basic relations and operations for CIFS with $r \in [0,1]$, but then has been expanded to $r \in [0,\sqrt{2}]$ to cover the whole region in the IFIT [11]. Some studies on CIFS have been conducted, including distance measures [11,14] and divergence measures for CIFS [20]. Other than that, some extensions of decision-making models under the CIFS environment have also been proposed recently, such as in technique for order preference by similarity to ideal solution (TOPSIS) [18,6], multiple criteria optimization and compromise solution (VIKOR) [19], the integration of analytic hierarchy process (AHP) and VIKOR [23] and a general multiple criteria decision making (MCDM) model [12]. In this paper, the concept of Pythagorean fuzzy set is investigated and compared with other types of uncertainty sets. We define Pythagorean fuzzy sub algebras in BCK-algebras and BCI-algebras and study their properties. A given Pythagorean fuzzy sub algebra is used to create a new Pythagorean fuzzy sub algebra. Also we obtain the intersection of two Pythagorean fuzzy sub algebras to be a Pythagorean fuzzy sub algebra is proved and an example is given to show that the union of two Pythagorean fuzzy sub algebras may not be a Pythagorean fuzzy sub algebra. The characterization of cut set is also used in Pythagorean fuzzy sub algebra. The homomorphic image and pre image of Pythagorean fuzzy sub algebra is discussed. It turns out that Pythagorean fuzzy sub algebra is a subclass of Fermatean fuzzy sub algebra

2. Preliminaries

In this section, we just recall the basic concepts of BCK/BCI algebras.

If a set X has a special element '0' and binary operation '$*$' satisfying the conditions:

$$\text{(BCI1)}: (\forall x, y, z \in X),\ \left(\left((x * y) * (x * z)\right) * (z * y) = 0\right),$$

$$\text{(BCI2)}: (\forall x, y, z \in X),\ \left(\left(x * (x * y)\right) * y = 0\right),$$

$$\text{(BCI3)}: (\forall x \in X),\ (x * x = 0),$$

$$\text{(BCI4)}: (\forall x, y \in X),\ (x * y = 0\ , y * x = 0\ \Rightarrow x = y),$$

then we say that X is a BCI-algebra.

If a BCI-algebra X satisfies the following identity:

(BCI5): $(\forall x \in X), (0 * x = 0)$, then X is called a BCK-algebra.

The order relation "$\leq$" in a BCK/BCI-algebra X is defined as follows:

$(\forall x, y \in X), (x \leq y \Leftrightarrow x * y = 0)$…………………………………(1)

Every BCK/BCI-algebra X satisfies the following conditions:

$(\forall x \in X), (0 * x = 0)$…..…………………………………………(2)

$(\forall x, y, z \in X),\ (x \leq y \Rightarrow x * z \leq y * z\ , z * y \leq z * x)$……………..(3)

$(\forall x, y, z \in X),\ \big((x * y) * z = (x * z) * y\big)$……………………………(4)

A non-empty subset 'A' of a BCK/BCI-algebra X is called a sub algebra of X if $x * y \in A$ for all $x, y \in A$.

Every ideal 'S' of a BCK/BCI-algebra X satisfies the next assertion,

$(\forall x, y \in X), (x \leq y,\ y \in S \Rightarrow x \in S)$……………………….………..(5)

Let X and Y be BCK/BCI-algebras. A mapping $\varphi: X \to Y$ is called a homomorphism if it satisfies: $(\forall x, y \in X), \big(\varphi(x * y) = \varphi(x) * \varphi(y)\big)$…..……………(6)

Let $\mu_A: X \to [0,1]$ and $\vartheta_A: X \to [0,1]$ be fuzzy sets in a set X. The structure $\Delta = \{(x, \mu_A(x), \vartheta_A(x))/x \in X\}$ is called

(i) an intuitionistic fuzzy set in X, if it satisfies

$(\forall x \in X), (0 \leq \mu_A(x) + \vartheta_A(x) \leq 1)$…………………..(7)

(ii) a Pythagorean fuzzy set in X, if it satisfies

$(\forall x \in X), \big(0 \leq {\mu_A}^2(x) + {\vartheta_A}^2(x) \leq 1\big)$………………..(8)

(iii) a (3,2)-fuzzy set in X, if it satisfies

$(\forall x \in X), \big(0 \leq {\mu_A}^3(x) + {\vartheta_A}^2(x) \leq 1\big)$………………..(9)

(iv) a square root (SR) fuzzy set in X, if it satisfies

$(\forall x \in X), \big(0 \leq {\mu_A}^2(x) + \sqrt{\vartheta_A(x)} \leq 1\big)$………………..(10)

(v) a cube root (CR) fuzzy set in X, if it satisfies

$(\forall x \in X), \big(0 \leq {\mu_A}^3(x) + \sqrt[3]{\vartheta_A(x)} \leq 1\big)$………………..(11)

3. Pythagorean fuzzy set

Definition-3.1: Let $\mu_\Delta: X \to [0,1]$ and $\vartheta_\Delta: X \to [0,1]$ be two fuzzy sets in a set X. Let $(\forall x \in X), \big(0 \leq {\mu_\Delta}^2(x) + {\vartheta_\Delta}^2(x) \leq 1\big)$. Then the structure $P = \{\langle x, \mu_\Delta(x), \vartheta_\Delta(x)\rangle/x \in X\}$ is called the Pythagorean fuzzy set in X.

In what follows, we apply the notations ${\mu_\Delta}^2(x)$ and ${\vartheta_\Delta}^2(x)$ instead of $\big(\mu_\Delta(x)\big)^2$ and $(\vartheta_\Delta(x))^2$, respectively and the Pythagorean fuzzy set on X and is simply denoted by $P = (X, \mu_\Delta, \vartheta_\Delta)$. The collection of Pythagorean fuzzy sets on X is denoted by ${F_2}^2(X)$.

Example 3.2: Let $X = \{0, l, m, n, r\}$ be the set and define fuzzy sets $\mu_\Delta: X \to [0,1]$ and $\vartheta_\Delta: X \to [0,1]$ as follows:

X	0	l	m	n	r
μ_Δ	0.93	0.74	0.92	0.55	0.67
ϑ_Δ	0.87	0.41	0.74	0.65	0.57

Then $P = (X, \mu_\Delta, \vartheta_\Delta)$ is a $(5,3)$-fuzzy set on X if $3 \geq 9$. But it is not a $(5,3)$-fuzzy set on X for $3 \leq 8$ because $(0.93)^5 + (0.87)^3 = 1.3542 > 1$.

Example 3.3: Consider the $(5,3)$-fuzzy set $P = (X, \mu_\Delta, \vartheta_\Delta)$ on X for $3 \geq 9$ in previous example. It is not intuitionistic fuzzy set because of $\mu_\Delta(0) + \vartheta_\Delta(0) = 0.93 + 0.87 = 1.8 > 1$.

Since ${\mu_\Delta}^2(m) + {\vartheta_\Delta}^2(m) = (0.92)^2 + (0.74)^2 = 1.394 > 1$, we know that $P = (X, \mu_\Delta, \vartheta_\Delta)$ is not a (2, 2)-fuzzy set on X.

Because of ${\mu_\Delta}^3(0) + {\vartheta_\Delta}^3(0) = (0.93)^3 + (0.87)^3 = 1.4629 > 1$, we know that $P = (X, \mu_\Delta, \vartheta_\Delta)$ is not a Fermatean fuzzy set on X.

Finally $P = (X, \mu_\Delta, \vartheta_\Delta)$ is not a 5-Pythogorean fuzzy set on X, since ${\mu_\Delta}^5(0) + {\vartheta_\Delta}^5(0) = (0.93)^5 + (0.87)^5 = 1.194 > 1$.

Definition 3.4 : We define a binary relation '$\lesssim$' and the equality '$=$' in ${F_2}^2(X)$ as follows

$P_1 \lesssim P_2 \Leftrightarrow \mu_{\Delta_1} \leq \mu_{\Delta_2}, \vartheta_{\Delta_1} \geq \vartheta_{\Delta_2}$....................................(1)

$P_1 = P_2 \Leftrightarrow \mu_{\Delta_1} = \mu_{\Delta_2}, \vartheta_{\Delta_1} = \vartheta_{\Delta_2}$....................................(2)

for all $P_1 = \left(X, \mu_{\Delta_1}, \vartheta_{\Delta_1}\right)$, $P_2 = \left(X, \mu_{\Delta_2}, \vartheta_{\Delta_2}\right)$ and $P_1 \neq P_2$. It is clear that $\left({F_2}^2(X), \lesssim\right)$ is a partially ordered set.

Definition 3.5 : For all $P_1 = \left(X, \mu_{\Delta_1}, \vartheta_{\Delta_1}\right)$ and $P_2 = \left(X, \mu_{\Delta_2}, \vartheta_{\Delta_2}\right) \in {F_2}^2(X)$, we define the union and the intersection as follows:

$P_1 \cup P_2 = \left(X, \mu_{\Delta_1} \cup \mu_{\Delta_2}, \vartheta_{\Delta_1} \cap \vartheta_{\Delta_2}\right)$.......................(3)

$P_1 \cap P_2 = \left(X, \mu_{\Delta_1} \cap \mu_{\Delta_2}, \vartheta_{\Delta_1} \cup \vartheta_{\Delta_2}\right)$.......................(4)

Where

$$\mu_{\Delta_1} \cup \mu_{\Delta_2} : X \to [0,1],\ x \to \max\left\{\mu_{\Delta_1}(x), \mu_{\Delta_2}(x)\right\}$$

$$\mu_{\Delta_1} \cap \mu_{\Delta_2} : X \to [0,1],\ x \to \min\left\{\mu_{\Delta_1}(x), \mu_{\Delta_2}(x)\right\}$$

$$\vartheta_{\Delta_1} \cup \vartheta_{\Delta_2} : X \to [0,1],\ x \to \max\left\{\vartheta_{\Delta_1}(x), \vartheta_{\Delta_2}(x)\right\}$$

$$\vartheta_{\Delta_1} \cap \vartheta_{\Delta_2} : X \to [0,1],\ x \to \min\left\{\vartheta_{\Delta_1}(x), \vartheta_{\Delta_2}(x)\right\}$$

It is clear that the union and intersection are associative binary operators in ${F_2}^2(X)$.

Example 3.6 : Let $X = \{0, l, m, n\}$ be a set and define fuzzy sets $P_1 = \left(X, \mu_{\Delta_1}, \vartheta_{\Delta_1}\right)$ and $P_2 = \left(X, \mu_{\Delta_2}, \vartheta_{\Delta_2}\right)$ on X by the tables below:

X	0	l	m	n
$\mu_{\Delta_1}(x)$	0.93	0.74	0.82	0.55

$\vartheta_{\Delta_1}(x)$	0.17	0.43	0.19	0.66

and

X	0	l	m	n
$\mu_{\Delta_2}(x)$	0.85	0.84	0.69	0.75
$\vartheta_{\Delta_2}(x)$	0.37	0.25	0.48	0.36

respectively. Then the union $P_1 \cup P_2$ of P_1 and P_2 is given below

X	0	l	m	n
$(\mu_{\Delta_1} \cup \mu_{\Delta_2})(x)$	0.93	0.84	0.82	0.75
$(\vartheta_{\Delta_1} \cup \vartheta_{\Delta_2})(x)$	0.17	0.25	0.19	0.36

Also, the intersection $P_1 \cap P_2$ of P_1 and P_2 is given by the table below

X	0	l	m	n
$(\mu_{\Delta_1} \cap \mu_{\Delta_2})(x)$	0.85	0.74	0.69	0.55
$(\vartheta_{\Delta_1} \cap \vartheta_{\Delta_2})(x)$	0.37	0.43	0.48	0.66

Proposition 3.7: Let $P_1 = (X, \mu_{\Delta_1}, \vartheta_{\Delta_1})$ and $P_2 = (X, \mu_{\Delta_2}, \vartheta_{\Delta_2}) \in {F_2}^2(X)$.

Then $P_1 \cap P_2 = P_2 \cap P_1$ (Commutative Law)

$P_1 \cup P_2 = P_2 \cup P_1$ (Commutative Law)

$(P_1 \cap P_2) \cup P_3 = P_2$ (Absorption Law)

$(P_1 \cup P_2) \cap P_3 = P_2$ (Absorption Law)

Proof: Straight forward.

Proposition 3.8: Every element of ${F_2}^2(X)$ is idempotent under the binary operations '$\cup$' and '$\cap$'.

Proof: Straight forward.

Theorem 3.9: $({F_2}^2(X), \cup, C_{01})$ and $({F_2}^2(X), \cap, C_{10})$ are commutative monoids where $C_{01} = (X, \tilde{0}, \tilde{1})$ and $C_{10} = (X, \tilde{1}, \tilde{0})$ with $\tilde{0}: X \to [0,1], x \mapsto \tilde{0}$ and $\tilde{1}: X \to [0,1], x \mapsto \tilde{1}$.

Proof: The proof is obvious.

Definition 3.10: The complement of $P = (X, \mu_\Delta, \vartheta_\Delta) \in {F_2}^2(X)$ is denoted by P^C $=(X, {\mu_\Delta}^C, {\vartheta_\Delta}^C)$ and is defined to be also Pythagorean fuzzy set $P^C = (X, \vartheta_\Delta, \mu_\Delta)$.

Example 3.11: Consider a Pythagorean fuzzy set $P = (X, \mu_\Delta, \vartheta_\Delta)$ and $X = \{0, 1, 2, 3\}$ which is defined by the following table

X	0	1	2	3
$\mu_\Delta(x)$	0.76	0.34	0.85	0.47
$\vartheta_\Delta(x)$	0.57	0.53	0.39	0.67

Then its complement $P^C = (X, \mu_\Delta{}^C, \vartheta_\Delta{}^C)$ is given as follows

X	0	1	2	3
$\mu_\Delta{}^C(x)$	0.57	0.53	0.39	0.67
$\vartheta_\Delta{}^C(x)$	0.76	0.34	0.85	0.47

Proposition 3.12: If $P_1 = (X, \mu_{\Delta_1}, \vartheta_{\Delta_1})$ and $P_2 = (X, \mu_{\Delta_2}, \vartheta_{\Delta_2}) \in F_2{}^2(X)$, then $(P_1 \cap P_2)^C = P_1{}^C \cup P_2{}^C$ and $(P_1 \cup P_2)^C = P_1{}^C \cap P_2{}^C$.

Proof: For a given $P_1 = (X, \mu_{\Delta_1}, \vartheta_{\Delta_1})$ and $P_2 = (X, \mu_{\Delta_2}, \vartheta_{\Delta_2}) \in F_2{}^2(X)$, we have

$$\begin{aligned}(P_1 \cap P_2)^C &= \left(X, (\mu_{\Delta_1} \cap \mu_{\Delta_2})^C, (\vartheta_{\Delta_1} \cup \vartheta_{\Delta_2})^C\right) \\ &= (X, \vartheta_{\Delta_1} \cup \vartheta_{\Delta_2}, \mu_{\Delta_1} \cap \mu_{\Delta_2}) \\ &= (X, \vartheta_{\Delta_1}, \mu_{\Delta_1}) \cup (X, \vartheta_{\Delta_2}, \mu_{\Delta_2}) \\ &= (X, \mu_{\Delta_1}{}^C, \vartheta_{\Delta_1}{}^C) \cup (X, \mu_{\Delta_2}{}^C, \vartheta_{\Delta_2}{}^C) \\ &= P_1{}^C \cup P_2{}^C\end{aligned}$$

The same way induces $(P_1 \cup P_2)^C = P_1{}^C \cap P_2{}^C$.

4. Pythagorean fuzzy sub algebras of BCK/BCI-algebras

In what follows, let X represent the BCK-algebra or BCI-algebra unless otherwise specified.

Definition 4.1`: A Pythagorean fuzzy set $P = (X, \mu_\Delta, \vartheta_\Delta)$ is called a Pythagorean fuzzy sub algebra of X if it satisfies

$$(\forall x, y \in X),\ \mu_\Delta{}^2(x * y) \geq \min\{\mu_\Delta{}^2(x), \mu_\Delta{}^2(y)\}$$
$$\vartheta_\Delta{}^2(x * y) \leq \max\{\vartheta_\Delta{}^2(x), \vartheta_\Delta{}^2(y)\}.$$

Example 4.2: Let $X = \{0, 1, 2, 3\}$ be the set with binary operation '$*$' in the table

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	1

2	2	1	0	2
3	3	3	3	0

Lemma 4.3: Every Pythagorean fuzzy sub algebra $P = (X, \mu_\Delta, \vartheta_\Delta)$ satisfies:

$(\forall x \in X),\ \left(\mu_\Delta{}^2(0) \geq \mu_\Delta{}^2(x), \vartheta_\Delta{}^2(0) \leq \vartheta_\Delta{}^2(x)\right)$.

Proof: From the definition-4.1, we have the following

$\mu_\Delta{}^2(0) = \mu_\Delta{}^2(x * x) \geq \min\{\mu_\Delta{}^2(x), \mu_\Delta{}^2(x)\} = \mu_\Delta{}^2(x)$,

$\vartheta_\Delta{}^2(0) = \vartheta_\Delta{}^2(x * x) \leq \max\{\vartheta_\Delta{}^2(x), \vartheta_\Delta{}^2(x)\} = \vartheta_\Delta{}^2(x)$ for all $x \in X$.

Theorem 4.4 : If $P = (X, \mu_\Delta, \vartheta_\Delta)$ is a Pythagorean fuzzy sub algebra of X, then the set $X_P = \{x \in X / \mu_\Delta{}^2(x) = \mu_\Delta{}^2(0), \vartheta_\Delta{}^2(x) = \vartheta_\Delta{}^2(0)\}$ is a sub algebra of X.

Proof: If $x, y \in X_P$, then $\mu_\Delta{}^2(x) = \mu_\Delta{}^2(0)$, $\mu_\Delta{}^2(y) = \mu_\Delta{}^2(0)$, $\vartheta_\Delta{}^2(x) = \vartheta_\Delta{}^2(0)$, $\vartheta_\Delta{}^2(y) = \vartheta_\Delta{}^2(0)$. It follows from the definition-4.1 that

$\mu_\Delta{}^2(x * y) \geq \min\{\mu_\Delta{}^2(x), \mu_\Delta{}^2(y)\} = \mu_\Delta{}^2(0)$ and

$\vartheta_\Delta{}^2(x * y) \leq \max\{\vartheta_\Delta{}^2(x), \vartheta_\Delta{}^2(y)\} = \vartheta_\Delta{}^2(0)$.

By combining this and previous lemma we derive $\mu_\Delta{}^2(x * y) = \mu_\Delta{}^2(0)$ and $\vartheta_\Delta{}^2(x * y) = \vartheta_\Delta{}^2(0)$, so $x * y \in X_P$. Hence X_P is a sub algebra of X. Hence the proof.

Given a Pythagorean fuzzy set $P = (X, \mu_\Delta, \vartheta_\Delta)$ and define a new Pythagorean fuzzy set

$P^* = (X, \mu_\Delta{}^*, \vartheta_\Delta{}^*)$ on X as follows $\mu_\Delta{}^*: X \to [0,1],\ x \mapsto \frac{\mu_\Delta(x)}{\sup\{\mu_\Delta(x)/x \in X\}}$,

$\vartheta_\Delta{}^*: X \to [0,1],\ x \mapsto \frac{\vartheta_\Delta(x)}{\inf\{\vartheta_\Delta(x)/x \in X\}}$ where $\inf\{\vartheta_\Delta(x)/x \in X\} \neq 0$.

Theorem 4.5: If $P = (X, \mu_\Delta, \vartheta_\Delta)$ is a Pythagorean fuzzy sub algebra of X with $\vartheta_\Delta(0) \neq 0$, then

$P^* = (X, \mu_\Delta{}^*, \vartheta_\Delta{}^*)$ is a Pythagorean fuzzy sub algebra of X.

Proof: If $P = (X, \mu_\Delta, \vartheta_\Delta)$ is a Pythagorean fuzzy sub algebra of X, then

$\sup\{\mu_\Delta(x)/x \in X\} = \mu_\Delta(0)$ and $\inf\{\vartheta_\Delta(x)/x \in X\} = \vartheta_\Delta(0) \neq 0$.

Then we have

$$\mu_\Delta{}^2(x * y) = \left(\frac{\mu_\Delta(x * y)}{\sup\{\mu_\Delta(x * y)/x * y \in X\}}\right)^2 = \left(\frac{\mu_\Delta(x * y)}{\mu_\Delta(0)}\right)^2 = \frac{\mu_\Delta{}^2(x * y)}{\mu_\Delta{}^2(0)}$$

$$\geq \frac{1}{\mu_\Delta{}^2(0)} \min\{\mu_\Delta{}^2(x), \mu_\Delta{}^2(y)\}$$

$$= \min\left\{\frac{\mu_\Delta{}^2(x)}{\mu_\Delta{}^2(0)}, \frac{\mu_\Delta{}^2(y)}{\mu_\Delta{}^2(0)}\right\} = \min\left\{\left(\frac{\mu_\Delta(x)}{\mu_\Delta(0)}\right)^2, \left(\frac{\mu_\Delta(y)}{\mu_\Delta(0)}\right)^2\right\}$$

$$= \min\{\mu_\Delta{}^{*2}(x), \mu_\Delta{}^{*2}(y)\}$$ and

$$\vartheta_\Delta{}^2(x*y) = \left(\frac{\vartheta_\Delta(x*y)}{\inf\{\vartheta_\Delta(x*y)/x*y \in X\}}\right)^2 = \left(\frac{\vartheta_\Delta(x*y)}{\vartheta_\Delta(0)}\right)^2 = \frac{\vartheta_\Delta{}^2(x*y)}{\vartheta_\Delta{}^2(0)}$$

$$\leq \frac{1}{\vartheta_\Delta{}^2(0)} \max\{\vartheta_\Delta{}^2(x), \vartheta_\Delta{}^2(y)\}$$

$$= \max\left\{\frac{\vartheta_\Delta{}^2(x)}{\vartheta_\Delta{}^2(0)}, \frac{\vartheta_\Delta{}^2(y)}{\vartheta_\Delta{}^2(0)}\right\} = \min\left\{\left(\frac{\vartheta_\Delta(x)}{\vartheta_\Delta(0)}\right)^2, \left(\frac{\vartheta_\Delta(y)}{\vartheta_\Delta(0)}\right)^2\right\}$$

$$= \max\{\vartheta_\Delta{}^{*2}(x), \vartheta_\Delta{}^{*2}(y)\}$$

for all $x, y \in X$. So $P^* = (X, \mu_\Delta{}^*, \vartheta_\Delta{}^*)$ is a Pythagorean fuzzy sub algebra of X.

Theorem 4.6: If P_1 and P_2 are Pythagorean fuzzy sub algebras of X, then their intersection $P_1 \cap P_2$ is also a Pythagorean fuzzy sub algebra of X.

Proof: For every $x \in X$, we have

$$(\mu_{\Delta_1} \cap \mu_{\Delta_2})^2(x*y) = \min\{\mu_{\Delta_1}{}^2(x*y), \mu_{\Delta_2}{}^2(x*y)\}$$

$$\geq \min\left\{\min\{\mu_{\Delta_1}{}^2(x), \mu_{\Delta_1}{}^2(y)\}, \min\{\mu_{\Delta_2}{}^2(x), \mu_{\Delta_2}{}^2(y)\}\right\}$$

$$= \min\left\{\min\{\mu_{\Delta_1}{}^2(x), \mu_{\Delta_2}{}^2(x)\}, \min\{\mu_{\Delta_1}{}^2(y), \mu_{\Delta_2}{}^2(y)\}\right\}$$

$$= \min\left\{(\mu_{\Delta_1} \cap \mu_{\Delta_2})^2(x), (\mu_{\Delta_1} \cap \mu_{\Delta_2})^2(y)\right\} \text{ and}$$

$$(\vartheta_{\Delta_1} \cup \vartheta_{\Delta_2})^2(x*y) = \max\{\vartheta_{\Delta_1}{}^2(x*y), \vartheta_{\Delta_2}{}^2(x*y)\}$$

$$\leq \max\left\{\max\{\vartheta_{\Delta_1}{}^2(x), \vartheta_{\Delta_1}{}^2(y)\}, \max\{\vartheta_{\Delta_2}{}^2(x), \vartheta_{\Delta_2}{}^2(y)\}\right\}$$

$$= \max\left\{\max\{\vartheta_{\Delta_1}{}^2(x), \vartheta_{\Delta_2}{}^2(x)\}, \max\{\vartheta_{\Delta_1}{}^2(y), \vartheta_{\Delta_2}{}^2(y)\}\right\}$$

$$= \max\left\{(\vartheta_{\Delta_1} \cup \vartheta_{\Delta_2})^2(x), (\vartheta_{\Delta_1} \cup \vartheta_{\Delta_2})^2(y)\right\}$$

Then, $P_1 \cap P_2$ is a Pythagorean fuzzy sub algebra of X.

The following example shows that the union of two Pythogorean fuzzy sub algebras may not be a Pythogorean fazzy sub algebra.

Example 4.7: Let $X = \{0, 1, 2, 3\}$ be the set with binary operation '$*$' in the following table:

$*$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

$1 * 2 = 2 * 1 = 3.$

$3 * 2 = 2 * 3 = 1.$

Thus, X is BCI-algebra.

Let us define $P_1 = (X, \mu_{\Delta_1}, \vartheta_{\Delta_1})$, $P_2 = (X, \mu_{\Delta_2}, \vartheta_{\Delta_2}) \in {F_2}^2(X)$ in the following tables below, respectively

X	0	1	2	3
$\mu_{\Delta_1}(x)$	0.73	0.61	0.49	0.48
$\vartheta_{\Delta_1}(x)$	0.21	0.53	0.30	0.52

and

X	0	1	2	3
$\mu_{\Delta_2}(x)$	0.73	0.51	0.53	0.52
$\vartheta_{\Delta_2}(x)$	0.24	0.56	0.63	0.63

Then P_1 and P_2 are Pythagorean fuzzy sub algebras of X.

Then union $P_1 \cup P_2$ is calculated as follows.

X	0	1	2	3
$(\mu_{\Delta_1} \cup \mu_{\Delta_2})(x)$	0.73	0.61	0.53	0.52
$(\vartheta_{\Delta_1} \cap \vartheta_{\Delta_2})(x)$	0.21	0.53	0.30	0.52

and it is not Pythagorean fuzzy sub algebra of X. Because of

$$(\mu_{\Delta_1} \cup \mu_{\Delta_2})^2(3 * 1) = (\mu_{\Delta_1} \cup \mu_{\Delta_2})^2(2)$$

$$(\mu_{\Delta_1} \cup \mu_{\Delta_2})^2(2) = \max\{{\mu_{\Delta_1}}^2(2), {\mu_{\Delta_2}}^2(2)\}$$

$$= \max\{(0.49)^2, (0.53)^2\}$$

$$(0.53)^2 \ngeq (0.61)^2$$

$$= \min\left\{(\mu_{\Delta_1} \cup \mu_{\Delta_2})^2(3), (\mu_{\Delta_1} \cup \mu_{\Delta_2})^2(1)\right\} \text{ and}$$

$$(\vartheta_{\Delta_1} \cap \vartheta_{\Delta_2})^2(2 * 1) = (\vartheta_{\Delta_1} \cap \vartheta_{\Delta_2})^2(3)$$

$$(\vartheta_{\Delta_1} \cap \vartheta_{\Delta_2})^2(3) \quad = \min\{{\vartheta_{\Delta_1}}^2(3), {\vartheta_{\Delta_2}}^2(3)\}$$

$$= \min\{(0.52)^2, (0.63)^2\}$$

$$(0.52)^2 \nleq (0.30)^2$$

$$= \max\left\{(\vartheta_{\Delta_1} \cap \vartheta_{\Delta_2})^2(2), (\vartheta_{\Delta_1} \cap \vartheta_{\Delta_2})^2(1)\right\}$$

Definition 4.8: Let $P = (X, \mu_\Delta, \vartheta_\Delta) \in {F_2}^2(X)$. For every $(\alpha, \beta) \in [0,1] \times [0,1]$ with

$0 \leq \alpha^2 + \beta^2 \leq 1, \;\; P_{(\alpha,\beta)} = P_\alpha \cap P_\beta$……………………….(1)

which is called a cut set of P when

$P_\alpha = \{x \in X \,/ {\mu_\Delta}^2(x) \geq \alpha\}$ and $P_\beta = \{x \in X \,/ {\vartheta_\Delta}^2(x) \leq \beta\}$.

Proposition 4.9: Let $P = (X, \mu_\Delta, \vartheta_\Delta)$, $Q = (X, \delta_\Delta, \gamma_\Delta) \in {F_2}^2(X)$. Then

$P \leq Q \Rightarrow P_{(\alpha,\beta)} \subseteq Q_{(\alpha,\beta)}$………………………………………………(2)

$(\forall(m.n) \in [0,1] \times [0,1]), \left(m \leq \alpha, n \geq \beta \Rightarrow P_{(\alpha,\beta)} \subseteq P_{(m,n)}\right)$.

Proof: Assume that $P \leq Q$ and let $x \in P_{(\alpha,\beta)}$. Then $\mu_\Delta \leq \delta_\Delta$ and $\vartheta_\Delta \geq \gamma_\Delta$.

That is $\mu_\Delta(x) \leq \delta_\Delta(x)$ and $\vartheta_\Delta(x) \geq \gamma_\Delta(x)$ for all $x \in X$.

It follows that $\alpha \leq {\mu_\Delta}^2(x) \leq {\delta_\Delta}^2(x)$ and $\beta \geq {\vartheta_\Delta}^2(x) \geq {\gamma_\Delta}^2(x)$. Thus $x \in Q_{(\alpha,\beta)}$.

Hence the proof.

Now let $(m.n) \in [0,1] \times [0,1]$ be such that $m \leq \alpha, n \geq \beta$.

If $x \in P_{(\alpha,\beta)}$, then ${\mu_\Delta}^2(x) \geq \alpha \geq m$ and ${\vartheta_\Delta}^2(x) \leq \beta \leq n$. Then $x \in P_{(m,n)}$.

So $P_{(\alpha,\beta)} \subseteq P_{(m,n)}$.

Theorem 4.10: If $P = (X, \mu_\Delta, \vartheta_\Delta)$ is Pythagorean fuzzy sub algebra of X, then its cut set $P_{(\alpha,\beta)}$ is a sub algebra of X.

Proof: Let $x, y \in P_{(\alpha,\beta)}$. Then ${\mu_\Delta}^2(x) \geq \alpha$ and ${\mu_\Delta}^2(y) \geq \alpha$, ${\vartheta_\Delta}^2(x) \leq \beta$ and ${\vartheta_\Delta}^2(y) \leq \beta$.

It follows from (1) that

${\mu_\Delta}^2(x * y) \geq \min\{{\mu_\Delta}^2(x), {\mu_\Delta}^2(y)\} \geq \min\{\alpha, \alpha\} \geq \alpha$ and

${\vartheta_\Delta}^2(x * y) \leq \max\{{\vartheta_\Delta}^2(x), {\vartheta_\Delta}^2(y)\} \leq \max\{\beta, \beta\} \leq \beta$.

Thus $x * y \in P_{(\alpha,\beta)}$. So $P_{(\alpha,\beta)}$ is a sub algebra of X.

Theorem 4.11: For a given P, if its cut set $P_{(\alpha,\beta)}$ is a sub algebra of X for every $(\alpha, \beta) \in [0,1] \times [0,1]$ with $0 \leq \alpha^2 + \beta^2 \leq 1$, then $P = (X, \mu_\Delta, \vartheta_\Delta)$ is Pythagorean fuzzy sub algebra of X.

Proof: Assume that $P_{(\alpha,\beta)}$ is a sub algebra of X for every $(\alpha, \beta) \in [0,1] \times [0,1]$ with $0 \leq \alpha^2 + \beta^2 \leq 1$. For every $x, y \in X$, we put $\alpha_x = {\mu_\Delta}^2(x)$, $\beta_x = {\vartheta_\Delta}^2(x)$ and $\alpha_y = {\mu_\Delta}^2(y)$, $\beta_y = {\vartheta_\Delta}^2(y)$.

Then $x, y \in P_{(\alpha,\beta)}$ for $\alpha = \min\{\alpha_x, \alpha_y\}$ and $\beta = \max\{\beta_x, \beta_y\}$. Thus $x * y \in P_{(\alpha,\beta)}$.

It follows that

${\mu_\Delta}^2(x * y) \geq \alpha = \min\{\alpha_x, \alpha_y\} = \min\{{\mu_\Delta}^2(x), {\mu_\Delta}^2(y)\}$ and

${\vartheta_\Delta}^2(x * y) \leq \beta = \max\{\beta_x, \beta_y\} = \max\{{\vartheta_\Delta}^2(x), {\vartheta_\Delta}^2(y)\}$.

So $P = (X, \mu_\Delta, \vartheta_\Delta)$ is Pythogorean fuzzy sub algebra of X.

Definition 4.12: Let P and Q be Pythagorean fuzzy sets on X and Y repectively. Let $\varphi: X \to Y$ be a mapping from a set X to a set Y.

(i) The pre image of $Q = (Y, \delta_\Delta, \gamma_\Delta)$ under φ is defined to be Pythagorean fuzzy set $\varphi^{-1}(Q)$ on X where

$\varphi^{-1}(\delta_\Delta): X \to [0,1]$, $x \to \delta_\Delta(\varphi(x))$ and $\varphi^{-1}(\gamma_\Delta): X \to [0,1]$, $x \to \gamma_\Delta(\varphi(x))$.

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(ii) The image of $P = (X, \mu_\Delta, \vartheta_\Delta)$ under φ is defined to be Pythagorean fuzzy set where

$$\varphi(\mu_\Delta): Y \to [0,1],\ y \to \begin{cases} \sup_{x\in\varphi^{-1}(y)} \mu_\Delta(x), & \text{if } \varphi^{-1}(y) \neq 0 \\ 0, & \text{otherwise} \end{cases}$$ and

$$\varphi(\vartheta_\Delta): Y \to [0,1],\ y \to \begin{cases} \inf_{x\in\varphi^{-1}(y)} \vartheta_\Delta(x), & \text{if } \varphi^{-1}(y) \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

Theorem 4.13: Let $\varphi: X \to Y$ be a homomorphism of BCK/BCI-algebras. If Q is Pythagorean fuzzy sub algebra of Y, then its pre image $\varphi^{-1}(Q)$ under φ is Pythagorean fuzzy sub algebra of X.

Proof: For every $x, y \in X$, we have

$$\begin{aligned}
\varphi^{-1}(\delta_\Delta)^2(x * y) &= \left(\varphi^{-1}(\delta_\Delta)(x * y)\right)^2 \\
&= \left(\delta_\Delta\big(\varphi(x * y)\big)\right)^2 = \left(\delta_\Delta\big(\varphi(x) * \varphi(y)\big)\right)^2 \\
&= {\delta_\Delta}^2\big(\varphi(x) * \varphi(y)\big) \\
&\geq \min\{{\delta_\Delta}^2\big(\varphi(x)\big), {\delta_\Delta}^2\big(\varphi(y)\big)\} \\
&= \min\left\{\big(\delta_\Delta\varphi(x)\big)^2, \big(\delta_\Delta\varphi(y)\big)^2\right\} \\
&= \min\left\{\big(\varphi^{-1}(\delta_\Delta)(x)\big)^2, \big(\varphi^{-1}(\delta_\Delta)(y)\big)^2\right\} \\
&= \min\{\varphi^{-1}(\delta_\Delta)^2(x), \varphi^{-1}(\delta_\Delta)^2(y)\} \text{ and}
\end{aligned}$$

$$\begin{aligned}
\varphi^{-1}(\gamma_\Delta)^2(x * y) &= \left(\varphi^{-1}(\gamma_\Delta)(x * y)\right)^2 \\
&= \left(\gamma_\Delta\big(\varphi(x * y)\big)\right)^2 = \left(\gamma_\Delta\big(\varphi(x) * \varphi(y)\big)\right)^2 \\
&= {\gamma_\Delta}^2\big(\varphi(x) * \varphi(y)\big) \\
&\leq \max\{{\gamma_\Delta}^2\big(\varphi(x)\big), {\gamma_\Delta}^2\big(\varphi(y)\big)\} \\
&= \max\left\{\big(\gamma_\Delta\varphi(x)\big)^2, \big(\gamma_\Delta\varphi(y)\big)^2\right\} \\
&= \max\left\{\big(\varphi^{-1}(\gamma_\Delta)(x)\big)^2, \big(\varphi^{-1}(\gamma_\Delta)(y)\big)^2\right\} \\
&= \max\{\varphi^{-1}(\gamma_\Delta)^2(x), \varphi^{-1}(\gamma_\Delta)^2(y)\}
\end{aligned}$$

Then $\varphi^{-1}(Q)$ is a Pythagorean fuzzy sub algebra of X.

Theorem 4.14 : Let $\varphi: X \to Y$ be an onto homomorphism of BCK/BCI-algebras. If P is a Pythagorean fuzzy sub algebra of X, then the image $\varphi(P)$ under φ is Pythagorean fuzzy sub algebra of Y.

Proof: For every $y_1, y_2 \in Y$, we have

$\{x_1, x_2 \in X / x_1 \in \varphi^{-1}(y_1), x_2 \in \varphi^{-1}(y_2)\} \subseteq \{x \in X / x \in \varphi^{-1}(y_1 * y_2)\}$.

Then we get

$$\begin{aligned}\varphi(\mu_\Delta)^2(y_1 * y_2) &= \big(\varphi(\mu_\Delta)(y_1 * y_2)\big)^2\\ &= (\sup\{\mu_\Delta(x)/x \in \varphi^{-1}(y_1 * y_2)\})^2\\ &\geq (\sup\{\mu_\Delta(x_1 * x_2)/x_1 \in \varphi^{-1}(y_1), x_2 \in \varphi^{-1}(y_2)\})^2\\ &= \min\{\sup\{{\mu_\Delta}^2(x_1)/x_1 \in \varphi^{-1}(y_1)\}, \sup\{{\mu_\Delta}^2(x_2)/x_2 \in \varphi^{-1}(y_2)\}\}\\ &= \min\{\varphi(\mu_\Delta)^2(y_1), \varphi(\mu_\Delta)^2(y_2)\} \text{ and}\end{aligned}$$

$$\begin{aligned}\varphi(\vartheta_\Delta)^2(y_1 * y_2) &= \big(\varphi(\vartheta_\Delta)(y_1 * y_2)\big)^2\\ &= (\inf\{\vartheta_\Delta(x)/x \in \varphi^{-1}(y_1 * y_2)\})^2\\ &\leq (\inf\{\vartheta_\Delta(x_1 * x_2)/x_1 \in \varphi^{-1}(y_1), x_2 \in \varphi^{-1}(y_2)\})^2\\ &= \inf\{{\vartheta_\Delta}^2(x_1 * x_2)/x_1 \in \varphi^{-1}(y_1), x_2 \in \varphi^{-1}(y_2)\}\\ &= \inf\{\max\{{\vartheta_\Delta}^2(x_1), {\vartheta_\Delta}^2(x_2)\}/x_1 \in \varphi^{-1}(y_1), x_2 \in \varphi^{-1}(y_2)\}\\ &= \max\left\{\inf\{{\vartheta_\Delta}^2(x_1)/x_1 \in \varphi^{-1}(y_1)\}, \inf\{{\vartheta_\Delta}^2(x_2)/x_2 \in \varphi^{-1}(y_2)\}\right\}\\ &= \max\{\varphi(\vartheta_\Delta)^2(y_1), \varphi(\vartheta_\Delta)^2(y_2)\}\end{aligned}$$

Thus $\varphi(P)$ is Pythagorean fuzzy sub algebra of Y.

Finally, we discuss the relationship between intuitionistic fuzzy sub algebra and Pythagorean fuzzy algebra.

Theorem 4.15: Every intuitionistic fuzzy sub algebra is Pythagorean fuzzy sub algebra.

Proof: Let $P = (X, \mu_\Delta, \vartheta_\Delta)$ be an intuitionistic fuzzy sub algebra of X. Then

$\mu_\Delta(x * y) \geq \min\{\mu_\Delta(x), \mu_\Delta(y)\}$ and $\vartheta_\Delta(x * y) \leq \max\{\vartheta_\Delta(x), \vartheta_\Delta(y)\}$ for all $x, y \in X$……..…(1)

We consider the following cases:

Case(i): $\mu_\Delta(x) \geq \mu_\Delta(y)$ and $\vartheta_\Delta(x) \geq \vartheta_\Delta(y)$,

Case(ii): $\mu_\Delta(x) \geq \mu_\Delta(y)$ and $\vartheta_\Delta(x) < \vartheta_\Delta(y)$,

Case(iii): $\mu_\Delta(x) < \mu_\Delta(y)$ and $\vartheta_\Delta(x) \geq \vartheta_\Delta(y)$,

Case(iv): $\mu_\Delta(x) < \mu_\Delta(y)$ and $\vartheta_\Delta(x) < \vartheta_\Delta(y)$.

Case(i) implies ${\mu_\Delta}^2(x) \geq {\mu_\Delta}^2(y)$ and ${\vartheta_\Delta}^2(x) \geq {\vartheta_\Delta}^2(y)$, then

$$\begin{aligned}{\mu_\Delta}^2(x * y) &= \big(\mu_\Delta(x * y)\big)^2\\ &\geq (\min\{\mu_\Delta(x), \mu_\Delta(y)\})^2\\ &= \min\{{\mu_\Delta}^2(x), {\mu_\Delta}^2(y)\} \text{ and}\end{aligned}$$

$$\begin{aligned}{\vartheta_\Delta}^2(x * y) &= \big(\vartheta_\Delta(x * y)\big)^2\\ &\leq (\max\{\vartheta_\Delta(x), \vartheta_\Delta(y)\})^2\\ &= \max\{{\vartheta_\Delta}^2(x), {\vartheta_\Delta}^2(y)\}, \text{ for all } x, y \in X.\end{aligned}$$

In the rest of cases, the condition equation (1) can be derived in the same way. Thus, $P = (X, \mu_\Delta, \vartheta_\Delta)$ is a Pythagorean fuzzy sub algebra of X.

The converse of above theorem may not be true. In fact, Pythagorean fuzzy sub algebra $P = (X, \mu_\Delta, \vartheta_\Delta)$ of X for all $(3,3) \in N \times N$ with $(3,3) \notin \{(1,1), (1,2), (2,1)\}$ in example (4.7) is not an intuitionistic fuzzy sub algebra of X because of

$$\mu_\Delta(3) + \vartheta_\Delta(3) = 0.52 + 0.63 = 1.15 > 1.$$

Conclusion: As per the sub class of intuitionistic fuzzy set, Pythagorean fuzzy set and (3,2)-fuzzy set, we introduced the notion of Pythagorean fuzzy set and applied it to BCK/BCI-algebras. We gave some operations for Pythagorean fuzzy set and investigated their properties. We introduce Pythagorean fuzzy sub algebra in BCK/BCI-algebras and investigated several properties. Using the given Pythagorean fuzzy sub algebra, we established a new Pythagorean fuzzy sub algebra. We proved that the intersection of two Pythagorean fuzzy sub algebras is also a Pythagorean fuzzy sub algebra and provided an example is given to the union of two Pythagorean fuzzy sub algebras may not be a Fermatean fuzzy sub algebra.

Also, we used the cut set to obtain the structures of Pythagorean fuzzy sub algebra. We show that intuitionistic fuzzy sub algebra is a sub classes of Pythagorean fuzzy sub algebra and consider the homomorphic image and pre image of Pythagorean fuzzy sub algebra.

Future work: The idea of this paper and the results obtained will be used for the study of various types of logical algebra in the future. And considering research on soft set theory and rough set theory etc. based on (3,2)-fuzzy set is also a subject of future research. It also attempts to explore the role of source in solving problems that includes uncertainty.

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