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Dharwad Characteristic Polynomial and Dharwad Energy of Graphs

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ABSTRACT:

Let G be a finite, undirected, simple graph with a set of vertices $V(G)$ and edge set $E(G)$. The Dharwad Matrix of G is a matrix of order $n \times n$ whose $(i,j)^{th}$entry is $\sqrt{\left(deg(v_i)\right)^3 + \left(deg(v_j)\right)^3}$ if v_i and v_j are adjacent and 0 if v_i and v_j are not adjacent. Let the eigen values of the Dharwad Matrix $A_D(G)$ be $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. These are the Dharwad characteristic polynomial's roots. The Dharwad energy $E_D(G)$ is the total of the absolute values of eigen values of $A_D(G)$. The Dharwad energy and characteristic polynomial for some specific graphs are found in this paper.

Keywords: Dharwad Matrix, Dharwad Characteristic Polynomial, Dharwad Energy.

Mathematics Subject Classification:
05C07,05C38,05C50,05C92

1. Introduction

This paper examines a finite, simple, undirected graph that has a set of vertices $V(G) = \{v_1, v_2, \ldots\ldots v_n\}$ and edge set $E(G)$. The notation $v_i \sim v_j$ means v_i and v_j are adjacent[1]. A vertex's degree is determined by how many other vertices are connected to it. Let $A(G)$ be the adjacency matrix of G with eigen values be $\rho_1 \geq \rho_2 \geq \ldots \geq \rho_n$. These are called eigen values of G and they form spectrum of G [2]. The total of the absolute values of the eigen values of $A(G)$ is the energy $E(G)$ of G. The structure of the adjacency matrix has a significant impact on a graph's spectrum. A graph's spectrum alone can be used to derive a number of potential drawbacks[3]. For instance, the graph's second biggest eigenvalue can provide some insight about the graph's extension and randomness[3]. Finding the energy of the molecular orbitals of π-electrons in Hückel molecular orbital theory is one of the primary uses of graph spectra in chemistry[3].More information and details about graph energy can be found in MajstoroviŇc et al. (2009), Gutman et al. (2009), Gutman (2001), and Gutman (2005). Numerous graph energies exist, including Randi´c energy (Alikhani and Ghanbari, 2015; Bozkurt and Bozkurt, 2013; Bozkurt et al. (2010), Das and Sorgun (2014), Gutman et al. (2014), Laplacian energy (Das et al. 2013), matching energy (Chen and Shi 2015; Ji et al. 2013), incidence energy (Bozkurt and Gutman 2013), and distance energy (Stevanovi¨c et al. 2013). Motivated by the Arithmetic-geometric energy[4] and Sombor energy[1] of specific graphs here we calculated Dharwad energy for some specific graphs.

Dharwad index[5] is defined as $D(G) = \sum_{v_i v_j \in E(G)} \sqrt{\left(deg(v_i)\right)^3 + \left(deg(v_j)\right)^3}$

A graph G's Dharwad Matrix is described as

$$A_D(G) = \left(a_{ij}\right)_{n\times n} = \begin{cases} \sqrt{\left(deg(v_i)\right)^3 + \left(deg(v_j)\right)^3} & if\ \ v_i \sim v_j \\ 0 & Otherwise \end{cases}$$

The eigen values of the Dharwad Matrix $A_D(G)$ be $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ which are the Dharwad characteristic polynomial's roots. $\emptyset_D(G,\lambda) = det\left(\lambda I - A_D(G)\right) = \prod_{i=1}^{n}(\lambda - \lambda_i)$. The Dharwad energy $E_D(G) = \sum_{i=1}^{n} |\lambda_i|$.

2. Results

Here, we calculate the Dharwad energy and characteristic polynomial for the complete graph, star graph, and complete bipartite graphs. Also determined the Dharwad characteristic polynomial of the path graph and the cycle graph.

Theorem 2.1

The Dharwad characteristic polynomial and the Dharwad energy of the complete graph K_n ; $n \geq 2$ are

$$\emptyset_D(K_n,\lambda) = \left(\lambda - \sqrt{2}\,(n-1)^{5/2}\right)\left(\lambda + \sqrt{2}\,(n-1)^{3/2}\right)^{n-1}$$

$$E_D(K_n) = 2\sqrt{2}\,(n-1)^{5/2}$$

Proof:

The Dharwad matrix of K_n is $\sqrt{2(n-1)^3}(J - I)$.

Therefore $\emptyset_D(K_n,\lambda) = det\left(\lambda I - \sqrt{2(n-1)^3}\,J + \sqrt{2(n-1)^3}\,I\right)$

$$= det\left(\left(\lambda + \sqrt{2(n-1)^3}\right) I - \sqrt{2(n-1)^3}\,J\right)$$

Since the eigen values of J_n are n and 0 (occurs once and $n-1$ times respectively), the eigen values of $\sqrt{2(n-1)^3}\,J_n$ are $n\sqrt{2(n-1)^3}$ and 0 (occurs once and $n-1$ times respectively). Therefore

$$\emptyset_D(K_n,\lambda) = \left(\lambda - \sqrt{2}\,(n-1)^{5/2}\right)\left(\lambda + \sqrt{2}\,(n-1)^{3/2}\right)^{n-1}$$

Since the eigen values are $\sqrt{2}\,(n-1)^{5/2}$ with multiplicity 1 and $-\sqrt{2}\,(n-1)^{3/2}$ with multiplicity $n-1$, we have

$$E_D(K_n) = 2\sqrt{2}\,(n-1)^{5/2}$$

Lemma 2.1

For a non-singular square matrix M, we have

$$det\begin{pmatrix} M & N \\ P & Q \end{pmatrix} = det(M)det(Q - PM^{-1}N)$$

where M^{-1} and $det(M)$ are the inverse and determinant of the matrix M.[6]

Theorem 2.2

The Dharwad characteristic polynomial and the Dharwad energy of the star graph $S_n = K_{1,n-1}$; $n \geq 2$ are

$$\emptyset_D(S_n, \lambda) = \lambda^{n-2}\left(\lambda^2 - (n-1)(n^3 - 3n^2 + 3n)\right)$$

$$E_D(S_n) = 2\sqrt{(n-1)(n^3 - 3n^2 + 3n)}$$

Proof:

The Dharwad matrix of $S_n = K_{1,n-1}$ is

$$A_D(S_n) = \sqrt{n^3 - 3n^2 + 3n}\begin{bmatrix} 0_{1\times 1} & J_{1\times n-1} \\ J_{n-1\times 1} & 0_{n-1\times n-1} \end{bmatrix}$$

We have

$$\emptyset_D(S_n, \lambda) = det\left(\lambda I - A_D(S_n)\right) =$$
$$det\begin{bmatrix} \lambda & -\sqrt{n^3 - 3n^2 + 3n}\, J_{1\times n-1} \\ -\sqrt{n^3 - 3n^2 + 3n}\, J_{n-1\times 1} & \lambda\, I_{n-1} \end{bmatrix}$$

Using Lemma 2.1, Dharwad characteristic polynomial of S_n is given by

$$\begin{aligned} \emptyset_D(S_n, \lambda) &= \lambda\, det\left(\lambda\, I_{n-1} - \sqrt{n^3 - 3n^2 + 3n} J_{n-1\times 1} \times \frac{1}{\lambda} \times \sqrt{n^3 - 3n^2 + 3n}\, J_{1\times n-1}\right) \\ &= \lambda^{2-n} det(\lambda^2 I_{n-1} - (n^3 - 3n^2 + 3n)\, J_{n-1}) \end{aligned}$$

The eigen values of J_{n-1} are $n-1$ and 0 (occurs once and $n-2$ times respectively), the eigen values of $(n^3 - 3n^2 + 3n)J_{n-1}$ are $(n-1)(n^3 - 3n^2 + 3n)$ and 0 (occurs once and $n-2$ times respectively). Therefore

$$\emptyset_D(S_n, \lambda) = \lambda^{n-2}\left(\lambda^2 - (n-1)(n^3 - 3n^2 + 3n)\right)$$

Since the eigen values are 0 with multiplicity $n-2$, $+\sqrt{(n-1)(n^3 - 3n^2 + 3n)}$ and $-\sqrt{(n-1)(n^3 - 3n^2 + 3n)}$, we have

$$E_D(S_n) = 2\sqrt{(n-1)(n^3 - 3n^2 + 3n)}$$

Theorem 2.3

The Dharwad characteristic polynomial and the Dharwad energy of the complete bipartite graph $K_{m,n}$; $m, n \neq 1$ are

$$\emptyset_D\left(K_{m,n}, \lambda\right) = \lambda^{m+n-2}\left(\lambda^2 - mn(m^3 + n^3)\right)$$

$$E_D\left(K_{m,n}\right) = 2\sqrt{mn(m^3 + n^3)}$$

Proof:

The Dharwad matrix of $K_{m,n}$ is

$$A_D(K_{m,n}) = \sqrt{m^3+n^3}\begin{bmatrix} 0_{m\times m} & J_{m\times n} \\ J_{n\times m} & 0_{n\times n}\end{bmatrix}$$

We have

$$\emptyset_D(K_{m,n},\lambda) = det\left(\lambda I - A_D(K_{m,n})\right)$$

$$= det\begin{bmatrix} \lambda I_m & -\sqrt{m^3+n^3}\,J_{m\times n} \\ -\sqrt{m^3+n^3}\,J_{n\times m} & \lambda I_n\end{bmatrix}$$

Using Lemma 2.1, Dharwad characteristic polynomial of $K_{m,n}$ is given by

$$\emptyset_D(K_{m,n},\lambda) = det(\lambda I_m)\ det\left(\lambda I_n - \sqrt{m^3+n^3}\,J_{n\times m} \times \frac{1}{\lambda} I_m \times \sqrt{m^3+n^3}\,J_{m\times n}\right)$$

$$= \lambda^m det\left(\lambda I_n - \frac{1}{\lambda}(m^3+n^3)\,m\,J_n\right)$$

$$= \lambda^{m-n} det(\lambda^2 I_n - m(m^3+n^3)\,J_n)$$

The eigen values of J_n are n and 0 (occurs once and $n-1$ times respectively), the eigen values of $m(m^3+n^3)J_n$ are $mn(m^3+n^3)$ and 0 (occurs once and $n-1$ times respectively). Therefore

$$\emptyset_D(K_{m,n},\lambda) = \lambda^{m+n-2}(\lambda^2 - mn(m^3+n^3))$$

Since the eigen values are 0 with multiplicity $m+n-2$, $+\sqrt{mn(m^3+n^3)}$ and $-\sqrt{mn(m^3+n^3)}$,

$$E_D(K_{m,n}) = 2\sqrt{mn(m^3+n^3)}$$

Theorem 2.4

The Dharwad characteristic polynomial of the path graph P_n; $n \geq 5$ satisfy

$$\emptyset_D(P_n,\lambda) = \lambda^2 det\,\Psi_{n-2} - 18\lambda\ det\,\Psi_{n-3} + 81 det\,\Psi_{n-4}$$

Proof:

For P_n we have

$$A_D(P_n) = \begin{pmatrix} 0 & 3 & 0 & 0 & & 0 & 0 & 0 \\ 3 & 0 & 4 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 4 & 0 & 4 & & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & & 0 & 0 & 0 \\ & \vdots & & & \ddots & & \vdots & \\ 0 & 0 & 0 & 0 & & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 4 & 0 & 3 \\ 0 & 0 & 0 & 0 & & 0 & 3 & 0 \end{pmatrix}$$

Let,

$$\Psi_k = \begin{pmatrix} \lambda & -4 & 0 & \cdots & 0 & 0 \\ -4 & \lambda & -4 & & 0 & 0 \\ & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & & \lambda & -4 \\ 0 & 0 & 0 & \cdots & -4 & \lambda \end{pmatrix}_{k\times k}$$

The Dharwad characteristic polynomial of P_n ,

$$\emptyset_D(P_n,\lambda) = det\big(\lambda I - A_D(P_n)\big)$$

$$= det\begin{pmatrix} \lambda & -3 & 0 & 0 & & 0 & 0 & 0 \\ -3 & \lambda & -4 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -4 & \lambda & -4 & & 0 & 0 & 0 \\ 0 & 0 & -4 & \lambda & & 0 & 0 & 0 \\ & \vdots & & & \ddots & & \vdots & \\ 0 & 0 & 0 & 0 & & \lambda & -4 & 0 \\ 0 & 0 & 0 & 0 & \cdots & -4 & \lambda & -3 \\ 0 & 0 & 0 & 0 & & 0 & -3 & \lambda \end{pmatrix}_{n\times n}$$

$$= \lambda\, det\begin{pmatrix} & \Psi_{n-2} & & 0 \\ & & & \vdots \\ & & & -3 \\ 0 & \ldots & -3 & \lambda \end{pmatrix} + 3det\begin{pmatrix} -3 & -4 & \cdots & 0 & 0 \\ \vdots & & \Psi_{n-3} & & \vdots \\ 0 & 0 & \cdots & -3 & \lambda \end{pmatrix}$$

$$= \lambda\left\{\lambda\, det\, \Psi_{n-2} + 3det\begin{pmatrix} & \Psi_{n-3} & & 0 \\ & & & \vdots \\ & & & 0 \\ 0 & \ldots & -4 & -3 \end{pmatrix}\right\} - 9\, det\begin{pmatrix} & \Psi_{n-3} & & 0 \\ & & & \vdots \\ & & & -3 \\ 0 & \ldots & -3 & \lambda \end{pmatrix}$$

$$= \lambda\{\lambda\, det\, \Psi_{n-2} - 9\, det\, \Psi_{n-3}\}$$
$$- 9\left\{\lambda\, det\, \Psi_{n-3} + 3det\begin{pmatrix} & \Psi_{n-4} & & 0 \\ & & & \vdots \\ & & & 0 \\ 0 & \ldots & -4 & -3 \end{pmatrix}\right\}$$

$$= \lambda\{\lambda\, det\, \Psi_{n-2} - 9\, det\, \Psi_{n-3}\} - 9\{\lambda\, det\, \Psi_{n-3} - 9\, det\, \Psi_{n-4}\}$$

$$= \lambda^2 det\, \Psi_{n-2} - 18\,\lambda\, det\, \Psi_{n-3} + 81\, det\, \Psi_{n-4}$$

Theorem 2.5

The Dharwad characteristic polynomial of the cycle graph C_n; $n \geq 3$ satisfy

$$\emptyset_D(C_n,\lambda) = \lambda\, det\, \Psi_{n-1} + 4\,\{-4\, det\, \Psi_{n-2} + (-1)^n(-4)^{n-1}\}$$
$$+ (-1)^{n+1}(-4)\{(-4)^{n-1} + (-1)^n(-4) det\, \Psi_{n-2}\}$$

Proof:

For C_n we have

$$A_D(C_n) = \begin{pmatrix} 0 & 4 & 0 & 0 & & 0 & 0 & 4 \\ 4 & 0 & 4 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 4 & 0 & 4 & & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & & 0 & 0 & 0 \\ & \vdots & & & \ddots & & \vdots & \\ 0 & 0 & 0 & 0 & & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 4 & 0 & 4 \\ 4 & 0 & 0 & 0 & & 0 & 4 & 0 \end{pmatrix}$$

Let,

$$\Psi_k = \begin{pmatrix} \lambda & -4 & 0 & \ldots & 0 & 0 \\ -4 & \lambda & -4 & & 0 & 0 \\ & \vdots & & \ddots & \vdots & \\ 0 & 0 & 0 & \ldots & \lambda & -4 \\ 0 & 0 & 0 & & -4 & \lambda \end{pmatrix}_{k\times k}$$

The Dharwad characteristic polynomial of C_n ,

$\emptyset_{SO-red}(C_n, \lambda) = det\big(\lambda I\ -\ A_{SO-red}(C_n)\big)$

$$= det\begin{pmatrix} \lambda & -4 & 0 & 0 & & 0 & 0 & -4 \\ -4 & \lambda & -4 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -4 & \lambda & -4 & & 0 & 0 & 0 \\ 0 & 0 & -4 & \lambda & & 0 & 0 & 0 \\ & \vdots & & & \ddots & & \vdots & \\ 0 & 0 & 0 & 0 & & \lambda & -4 & 0 \\ 0 & 0 & 0 & 0 & \cdots & -4 & \lambda & -4 \\ -4 & 0 & 0 & 0 & & 0 & -4 & \lambda \end{pmatrix}_{n\times n}$$

$$= \lambda\, det\ \Psi_{n-1} + 4\, det\begin{pmatrix} -4 & -4 & \dots & 0 \\ 0 & & & \\ \vdots & & \Psi_{n-2} & \\ -4 & & & \end{pmatrix} + (-1)^{n+1}(-4)det\begin{pmatrix} -4 & & & \\ 0 & & \Psi_{n-2} & \\ \vdots & & & \\ -4 & 0 & \dots & -4 \end{pmatrix}$$

$$= \lambda\, det\, \Psi_{n-1} + 4\,\{-4\, det\, \Psi_{n-2} + (-1)^n(-4)^{n-1}\} + (-1)^{n+1}(-4)\{(-4)^{n-1} + (-1)^n(-4)det\, \Psi_{n-2}\}.$$

3. Conclusion

In this paper, we obtained the Dharwad energy and characteristic polynomial for the complete graph, star graph, and complete bipartite graphs. Also calculated the Dharwad characteristic polynomial of the path graph and the cycle graph. The following ideas are prospective areas of interest for additional research:

- Dharwad energy and characteristic polynomial for specific graphs due to edge deletion.
- Determination of Dharwad energy and characteristic polynomial for other graph classes

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