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A MATHEMATICAL STUDY ON TWO SPECIES ECOLOGY PREY-PREDATOR WITH MIGRATION AND IMMIGRATION

Ch.Soma Shekar[1], V.Anand[2], Shwetha Srivastava[3] and B. Hari Prasad[4*]

[1]*Associate Professor, Malla Reddy College of Engineering and Technology, Telangana State, India*
[2]*Assistant Professor, Kakatiya Institute of Technology and Science, Telangana State, India*
[3]*Assistant Professor, RBVRR Women's College, Hyderabad, Telangana State, India*
[4]*Professor of Mathematics, Chaitanya (Deemed to be University), Telangana State, India*
[*]*Corresponding Author: E-mail: sumathi_prasad73@yahoo.com*

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Abstract
The purpose of this paper is to study on migration and immigration of prey-predator with limited resources for both the species. The system comprises of a prey (S_1), a predator (S_2) that survives upon S_1 with migration of prey and immigration of predator. The model equations of the system constitute a set of two simultaneous first order non-linear ordinary differential equations. The system would be stable if all the characteristic roots are negative, in case they are real, and have negative real parts, in case they are complex. All the four equilibrium points are identified and their stability discussed. The trajectories of solution curves for the equilibrium points are established. The global stability of linearized equations can be discussed by constructing a suitable Liaupnov's function. Further, the Runge-Kutta fourth order technique is used to compute the numerical solutions for the growth rate equations.
Keywords: Prey-Predator Dynamics, Equilibrium States, Liaupnov's Function, Numerical Solutions, Non-linear Ordinary Differential Equations.

1. Introduction

Ecology is the study of how living things interact with their surroundings. Ecologists research the interactions between organisms and their environments, particularly the number and distribution of different types of life on Earth. Ecology can be studied at various levels, from individual organisms to populations, communities, ecosystems, and the biosphere. It takes into account both living and non-living variables. Every level offers a different perspective on the intricate processes that make up our planet. In its most basic form, ecology is a mathematical study of population growth and fluctuations, i.e. plant population, animal population or other organic population. The mathematical analysis of ecological issues is not a new discipline; in fact, Lotka and Voltera were early pioneers who laid the groundwork for it. Scientists studying ecology have to deal with the dynamics of nature, such as the increase or decrease in population sizes of several plant and animal species. The [1] was implementation of mathematical modeling in ecological research on the example of the prevalence of odonates in Serbia. [2] presented the mathematical modeling of a new bio-inspired evolutionary search algorithm called Ecological Systems Algorithm. [3] show that faster-reproducing animals are more likely to have nonlinear and high-dimensional dynamics, supporting past ecological theory. [4] studies the birth and death processes in interactive random environments where the birth and death rates and the dynamics of the state of the environment are dependent on each other. [5] proposed and analyzed a detailed mathematical model describing the dynamics of a prey-predator model under the influence of an SIS infectious disease by using nonlinear differential equations. [6] highlight the effect of prey cannibalism on the interaction between predator and prey. Numerous ecologists made contributions to the expansion of this field of study[7, 8]. Numerous researchers have been drawn to study predator-prey systems, both continuous and discrete, because of their ability to display complicated dynamical behaviour [9-11]. Moreover, Numerous scholars [12–14] and mathematicians [14-17] have contributed significant viewpoints, worthwhile tenets, and intriguing applications to the field of mathematical ecology that allow for the analysis of the behaviours of various ecological models. In intercultural aspects, Hari Prasad investigated the local and global stability of numerous mathematical models (both discrete and continuous) of syn-ecology [18–24].

2. Basic Equations of the Model

The basic equation for the model is given by the following system of non-linear ordinary differential equations.

Equation for the growth rate of prey species (S_1):

$$dN_1/dt = a_1N_1 - a_{11}N_1^2 - a_{12}N_1N_2 - MN_1 \tag{1}$$

Equation for the growth rate of predator species (S_2):

$$dN_2/dt = a_2N_2 - a_{22}N_2^2 + a_{21}N_1N_2 + IN_2 \tag{2}$$

Notation adopted

$N_1(t)$: The population strength of prey species (S_1)

$N_2(t)$: The population strength of predator species (S_2)

t : Time instant

a_i : Natural growth rates of S_i, i = 1, 2

a_{ii} : Self inhibition coefficient of S_i, i = 1, 2

a_{12}, a_{21} : Interaction coefficients of S_1 due to S_2 and S_2 due to S_1

M, I : Migration and immigration coefficients of $S_1$and S_2

Further the variables N_1, N_2 are non-negative and the model parameters a_1, a_2, a_{11}, a_{22}, a_{12}, a_{21}, M, I are assumed to be non-negative constants.

3. Equilibrium States

The system under investigation has following three equilibrium states as $dN_i/dt = 0; i = 1,2$

Fully washed out state: $E_1 : \bar{N}_1 = 0, \bar{N}_2 = 0$

Prey washed out state: $E_2 : \bar{N}_1 = 0, \bar{N}_2 = k_2 + (I/a_{22})$

Predator washed out state: $E_3 : \bar{N}_1 = k_1 - \left(M/a_{11}\right),\ \bar{N}_2 = 0$

Normal steady state: $E_4 : \bar{N}_1 = \alpha_1/\alpha,\ \bar{N}_2 = \alpha_2/\alpha$

Where $\alpha_1 = a_{22}\left(a_1 - M\right) - a_{12}\left(a_2 + I\right), \alpha_2 = a_{21}\left(a_1 - M\right) + a_{11}\left(a_2 + I\right)$

and $\alpha = a_{11}a_{22} + a_{12}a_{21}$ (3)

4. Methodology

Stability of Equilibrium States

Let $N_i(t) = (N_1, N_2) = \bar{N}_i + U_i(t); i = 1,2$ where $U_i(t)$ is a small perturbation over the equilibrium point $N = (\bar{N}_1, \bar{N}_2)$ .The basic equations are quasi linearized to obtain the equations for the perturbed state as

$$du_1/dt = (a_1 - 2a_{11}\bar{N}_1 - a_{12}\bar{N}_2 - M)u_1 - a_{12}\bar{N}_1 u_2 \quad (4)$$

$$du_2/dt = a_{21}\bar{N}_2 u_1 + (a_2 - 2a_{22}\bar{N}_2 + a_{21}\bar{N}_1 + I)u_2 \quad (5)$$

The characteristic equation is $\left|A - \lambda I\right| = 0$ (6)

where $A = \begin{bmatrix} a_1 - 2a_{11}\bar{N}_1 - a_{12}\bar{N}_2 - M & -a_{12}\bar{N}_1 \\ a_{21}\bar{N}_2 & a_2 - 2a_{22}\bar{N}_2 + a_{21}\bar{N}_1 + I \end{bmatrix}$ (7)

The equilibrium state is stable, if all the roots of equation (6) are negative, in case they are real

or have the negative real parts , in case they are complex.

4.1 Stability of fully washed out state

The basic equations are quasi linearized, we get

$$du_1/dt = (a_1 - M)u_1;\ \ du_2/dt = (a_2 + I)u_2 \qquad (8)$$

The characteristic equation is $\left[\lambda - (a_1 - M)\right]\left[\lambda - (a_2 + I)\right] = 0$ (9)

The characteristic roots are $(a_1 - M)$ and $(a_2 + I)$. Since, one root $(a_2 + I) > 0$ is positive. Hence, E_1 is **unstable** and the solutions of the equations (8) are

$$u_1 = u_{10} e^{(a_1 - M)t};\ u_2 = u_{20} e^{(a_2 + I)t} \qquad (10)$$

Where u_{10} and u_{20} are the initial values of u_1 and u_2 respectively.

The trajectories in $u_1 - u_2$ plane are $\left(u_1/u_{10}\right)^{(a_2+I)} = \left(u_2/u_{20}\right)^{(a_1-M)}$

4.2 Stability of prey washed out state

The basic equations are linearized to obtain the equations as

$$du_1/dt = (a_1 - b_1)u_1;\ \ du_2/dt = a_{21}(k_2 + I/a_{22})u_1 + (a_2 + I)u_2 \qquad (11)$$

Where $b_1 = a_{12}[k_2 + (I/a_{22})] + M > 0$ (12)

The characteristic equation of (11) is $\left[\lambda - (a_1 - b_1)\right]\left[\lambda + (a_2 + I)\right] = 0$ (13)

The characteristic roots are $a_1 - b_1$ and $-(a_2 + I)$

Case (i): When $a_1 > b_1$, one root is positive. Hence the state E_2 is **unstable.**

Case (ii): When $a_1 < b_1$, both roots are negative. Hence, in this case the state is **stable.**

Case (iii): When $a_1 = b_1$, one root is zero and other is negative. Hence, E_2 is **neutrally stable.**

The solutions are given by

$$u_1 = u_{10} e^{(a_1 - b_1)t};\ u_2 = \left(u_{20} - \delta_{10} u_{10}\right) e^{-(a_2 + I)t} + \delta_{10} u_{10} e^{(a_1 - b_1)t} \qquad (14)$$

Where $\delta_{10} = a_{12}(a_2 + I)/a_{22}(a_1 - b_1 + a_2 + I)$ (15)

$u_2 = \left(u_{20} - \delta_{10} u_{10}\right)\left(u_1/u_{10}\right)^{b_2/(b_1 - a_1)} + \delta_{10} u_1$ are the trajectories in $u_1 - u_2$ plane.

4.3 Stability of Predator washed out state

The characteristic equation of the state is $\left[\lambda - (M - a_1)\right]\left[\lambda - (a_2 + I - d)\right] = 0$ (16)

Where $d = a_{21}(M - a_1)/a_{11}$ (17)

The characteristic roots are $M - a_1$ and $a_2 + I - d$

Case (i): If $M = a_1, a_2 + I = d; M = a_1, a_2 + I < d$ and $M < a_1, a_2 + I = d$ then the state is **neutrally stable.**

Case (ii): If $M < a_1$ and $a_2 + I < d$ then state is **stable** and the solutions are

$$u_1 = \left(u_{10} - \delta_2 u_{20}\right) e^{(M - a_1)t} + \delta_{20} u_{20} e^{(a_2 + I - d)t};\ u_2 = u_{20} e^{(a_2 + I - d)t} \tag{18}$$

$$\text{Where } \delta_{20} = d / [a_2 + I - (M - a_1)(a_{21} + a_{11})] \tag{19}$$

$u_1 = \left(u_{10} - \delta_{20} u_{20}\right)\left(u_2 / u_{20}\right)^{(M - a_1)/(a_2 + I - d)} + \delta_{20} u_2$ are the trajectories in $u_1 - u_2$ plane.

4.4 Stability of normal steady state

The characteristic equation is $\lambda^2 - \left(b_2 - d_2\right)\lambda - (\gamma_1 a_{21} / a_{12})\left(a_1 - M\right) = 0$ (20)

Let λ_1 and λ_2 be the zeros of the quadratic polynomial on the above equation (20).

When the roots λ_1, λ_2 noted to be negative. Hence, the normal steady state is **stable** and the system of equations yield the solutions,

$$u_1 = [(B + u_{10}\lambda_2)/(\lambda_2 - \lambda_1)]e^{\lambda_1 t} + [(B + u_{10}\lambda_1)/(\lambda_1 - \lambda_2)]e^{\lambda_2 t} \tag{21}$$

$$\begin{aligned} u_2 &= [(B + u_{10}\lambda_2)(a_{11}\bar{N}_1 + \lambda_1)/(\lambda_1 - \lambda_2)\, a_{12}\bar{N}_1]e^{\lambda_1 t} \\ &\quad + [(B + u_{10}\lambda_1)(a_{11}\bar{N}_1 + \lambda_2)/(\lambda_1 - \lambda_2) a_{12}\bar{N}_1]e^{\lambda_2 t} \end{aligned} \tag{22}$$

$$\text{Where } B = a_{12} u_{20} \bar{N}_1 + u_{10} a_{11} \bar{N}_1 \tag{23}$$

The trajectories in $u_1 - u_2$ plane are $[(u_1 - v_1 u_2)^{q v_1 + r}] / [(u_1 - v_2 u_2)^{q v_2 + r}] = c_1 u_2^{(1-q)(v_2 - v_1)}$

Here, v_1 and v_2 are roots of the quadratic equation $qV^2 + rV - s = 0$

Where, $q = a_{12}\bar{N}_2, r = a_{11}\bar{N}_1 - a_{22}\bar{N}_2, V = u_1 / u_2$ and c_1 is arbitrary constant.

5. Liaupnov's Function for Global Stability

The linearized equations (4) and (5) becomes

$$\frac{du_1}{dt} = -a_{11}\bar{N}_1 u_1 - a_{12}\bar{N}_1 u_2;\ \frac{du_2}{dt} = a_{21}\bar{N}_2 u_1 - 2a_{22}\bar{N}_2 u_2 \tag{24}$$

The characteristic equation is $\lambda^2 + p\lambda + q = 0$

where $p = a_{22}\overline{N_2} + a_{11}\overline{N_1} > 0, q = \left[a_{11}a_{22} + a_{12}a_{21}\right]\overline{N_1}\,\overline{N_2} > 0$

Define, $E(u_1,u_2)$ by $E(u_1,u_2) = \frac{1}{2}\left[au_1^{\,2} + 2bu_1u_2 + cu_2^{\,2}\right]$

Where, $a = \frac{1}{D}\left[\left(a_{21}\overline{N_2}\right)^2 + \left(a_{22}\overline{N_2}\right)^2 + \left\{a_{11}a_{22} + a_{12}a_{21}\right\}\overline{N_1}\,\overline{N_2}\right]$; $b = \frac{1}{D}\left(a_{11}a_{21} + a_{12}a_{22}\right)\overline{N_1}\,\overline{N_2}$

$c = \frac{1}{D}\left[\left(a_{11}\overline{N_1}\right)^2 + \left(a_{12}\overline{N_1}\right)^2 + \left\{a_{11}a_{22} + a_{12}a_{21}\right\}\right]\overline{N_1}\,\overline{N_2}$; $D = \left(a_{22}\overline{N_2} + a_{11}\overline{N_1}\right)\left(\left[a_{11}a_{22} + a_{12}a_{211}\right]\right)\overline{N_1}\,\overline{N_2}$

It is clear that $D > 0, a > 0$.

Also, $(ac - b^2) = \left(\frac{1}{D}\left[\left(a_{21}\overline{N_2}\right)^2 + \left(a_{22}\overline{N_2}\right)^2 + \left\{a_{11}a_{22} + a_{12}a_{21}\right\}\right]\overline{N_1}\,\overline{N_2}\right) \mathrm{X}$

$$\left(\frac{1}{D}\left[\left(a_{11}\overline{N_1}\right)^2 + \left(a_{12}\overline{N_1}\right)^2 + \left\{a_{11}a_{22} + a_{12}a_{21}\right\}\right]\overline{N_1}\,\overline{N_2}\right) - \left(\frac{1}{D}\left(a_{11}a_{21} + a_{12}a_{22}\right)\overline{N_1}\,\overline{N_2}\right)^2$$

$$= \frac{1}{D^2}\left\{\left[\left(a_{21}\overline{N_2}\right)^2 + \left(a_{22}\overline{N_2}\right)^2\right]\left\{a_{11}a_{22} + a_{12}a_{21}\right\} + \left\{\left[\left(a_{11}\overline{N_1}\right)^2 + \left(a_{12}\overline{N_1}\right)^2\right]\left\{a_{11}a_{22} + a_{12}a_{21}\right\}\right\}\right\}$$

$$+\left[\left(a_{21}\overline{N_2}\right)^2 + \left(a_{22}\overline{N_2}\right)^2\right]\left[\left(a_{11}\overline{N_1}\right)^2 + \left(a_{12}\overline{N_1}\right)^2\right] > 0 \Rightarrow D^2(ac - b^2) > 0$$

Therefore $E(u_1,u_2)$ is positive definite. Further, we find

$$\frac{\partial E}{\partial u_1}.\frac{du_1}{dt} + \frac{\partial E}{\partial u_2}.\frac{du_2}{dt} = \left(au_1 + bu_2\right)(-a_{11}\overline{N_1}u_1 - a_{12}\overline{N_1}u_2) + (bu_1 + cu_2)(a_{21}\overline{N_2}u_1 - a_{22}\overline{N_2}u_2)$$

Substituting the terms a,b,c we have $\frac{\partial E}{\partial u_1}.\frac{du_1}{dt} + \frac{\partial E}{\partial u_2}.\frac{du_2}{dt} < 0$ is negative definite.

This implies $E(u_1,u_2)$ exists as to linear system to Liapunov's function

Now, we discuss $E(u_1,u_2)$ as Liapunov's function to non linear system.

$$F_1(N_1,N_2) = N_1(a_1 - a_{11}N_1 - a_{12}N_2 - M);\; F_2(N_1,N_2) = N_2(a_2 - a_{22}N_2 + a_{21}N_1 + I)$$

Letting $N_1 = \overline{N_1} + u_1, N_2 = \overline{N_2} + u_2$,

$$\frac{du_1}{dt} = \left(\overline{N_1} + u_1\right)\left[a_1 - a_{11}\left(\overline{N_1} + u_1\right) - a_{12}\left(\overline{N_2} + u_2\right) - M\right]$$

$$= -a_{11}\overline{N_1}u_1 - a_{12}\overline{N_1}u_2 + f_1(u_1,u_2) = F_1\ (u_1,u_2)$$

$$\frac{du_2}{dt} = \left(\overline{N_2} + u_2\right)\left[a_2 - a_{22}\left(\overline{N_2} + u_2\right) + a_{21}\left(\overline{N_1} + u_1\right) + I\right]$$

$$= a_{21}\overline{N_2}u_1 - a_{22}\overline{N_2}u_2 + f_2(u_1,u_2) = F_2\ (u_1,u_2)$$

Now, $\frac{\partial E}{\partial u_1}.F_1 + \frac{\partial E}{\partial u_2}.F_2 = \frac{\partial E}{\partial u_1}.\frac{du_1}{dt} + \frac{\partial E}{\partial u_2}.\frac{du_2}{dt}$

$$= \left(au_1 + bu_2\right)(-a_{11}\overline{N_1}u_1 - a_{12}\overline{N_1}u_2 + f_1(u_1,u_2)) + +(bu_1 + cu_2)(a_{21}\overline{N_2}u_1 - a_{22}\overline{N_2}u_2 + f_2(u_1,u_2))$$

Introducing the polar coordinates $u_1 = r\cos\theta, u_2 = r\sin\theta$ and solving we get

Therefore $\frac{\partial E}{\partial u_1}.F_1 + \frac{\partial E}{\partial u_2}.F_2 < 0$ is negative definite.

Furthermore $E\left(u_1,u_2\right)$ is positive definite with reason $\frac{\partial E}{\partial u_1}.F_1 + \frac{\partial E}{\partial u_2}.F_2$ is negative definite with the conclusion that point of equilibrium is asymptotically "stable".

6. Numerical Study

Table: Parameter Values

Fig.No.	a_1	a_{11}	a_{12}	M	a_2	a_{22}	a_{21}	I	N_{10}	N_{20}	t^*
1	3.23	0.13	1.30	0.73	0.67	1.10	0.20	0.90	0.56	0.46	5.80
2	1.26	3.186	0.666	1.386	0.72	2.16	1.134	0.846	3.474	5.184	-
3	0.53	0.30	0.20	0.20	0.30	0.47	0.37	0.33	9.44	0.88	0.90
4	14.944	1.08	0.846	3.906	1.08	0.594	0.414	0.486	2.646	0.45	1.00
5	6.84	1.026	0.594	0.594	8.28	14.76	16.794	2.466	0.50	0.90	-
6	16.56	2.40	6.40	8.56	0.24	2.96	1.04	0.80	1.50	5.00	0.91
7	19.06	2.00	0.94	0.00	0.94	0.80	0.66	0.94	1.00	1.00	0.72
8	0.001	1.04	0.56	0.56	0.56	9.04	0.24	0.56	2.56	0.56	2.01

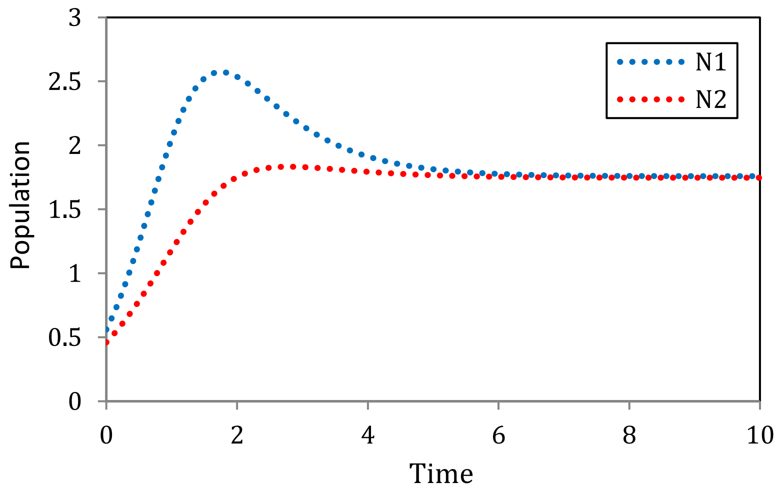

Figure 1: N_1=0.56, N_2=0.46

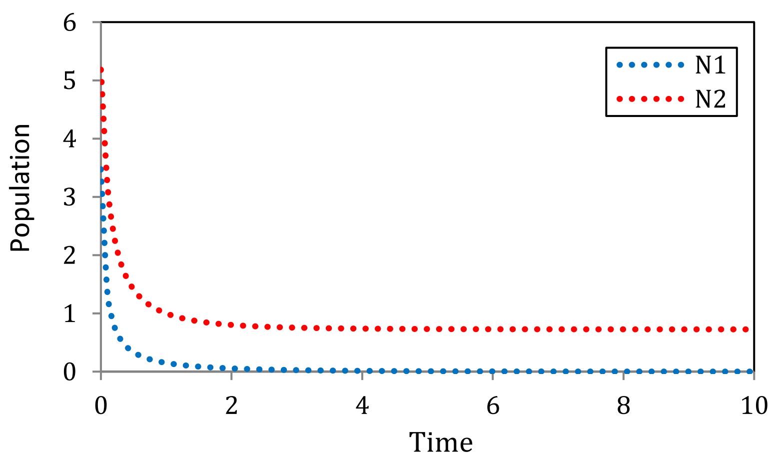

Figure 2: N_1=3.474, N_2=5.184

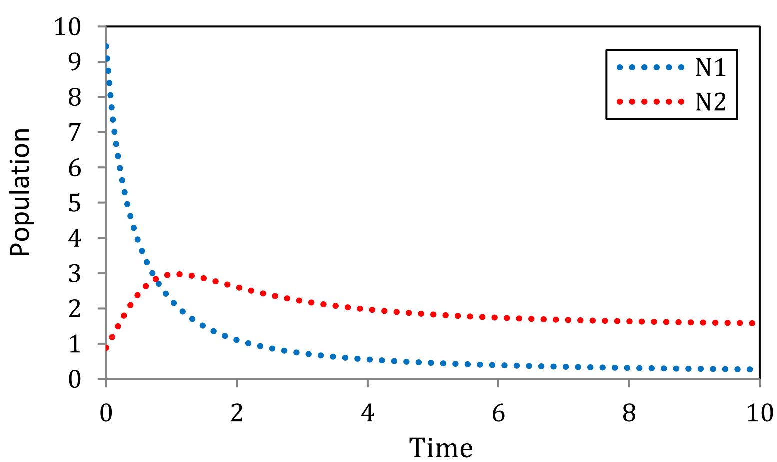

Figure 3: N_1=9.44, N_2=0.88

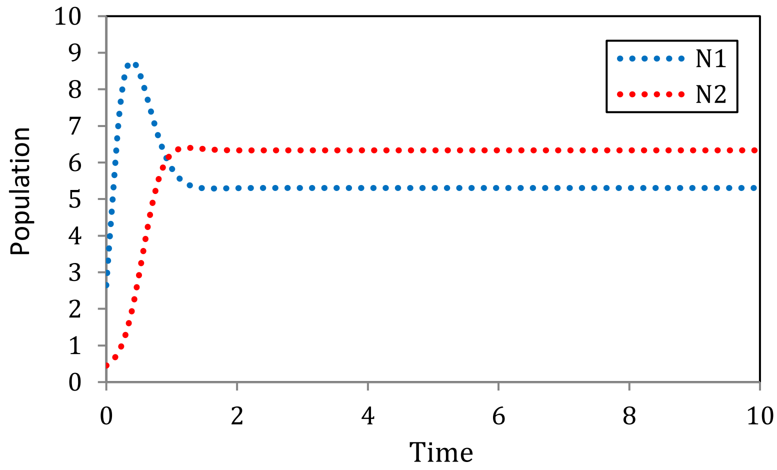

Figure 4: N_1=2.646, N_2=0.45

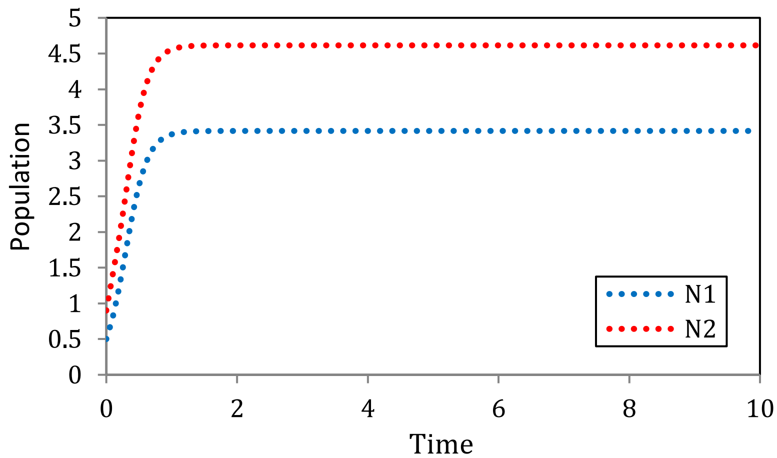

Figure 5: N_1=0.50, N_2=0.90

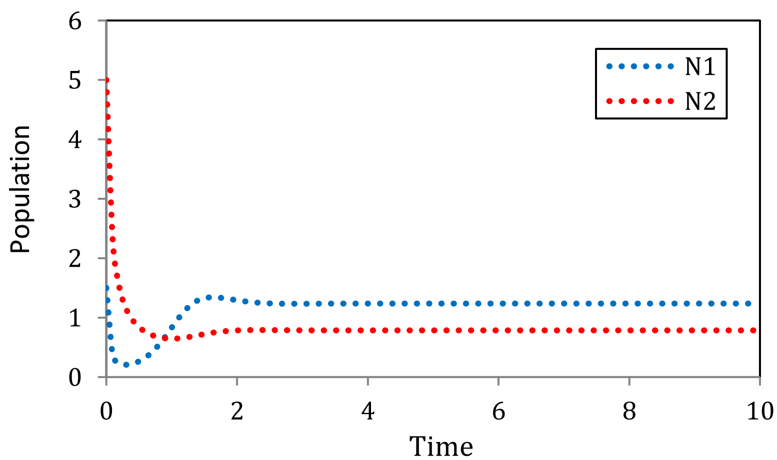

Figure 6: N_1=1.50, N_2=5.00

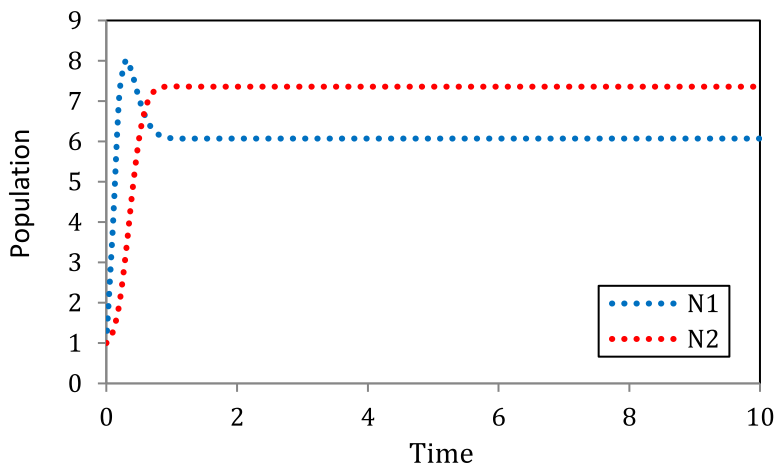

Figure 7: N_1=1.00, N_2=1.00

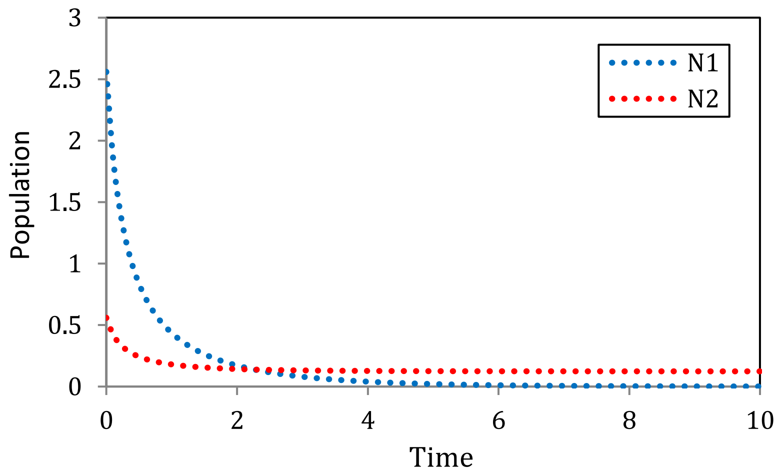

Figure 8: N_1=2.56, N_2=0.56

7. Discussions and Conclusion

Situation 1: In this case the immigration coefficient is greater than migration coefficient. Further, the prey and predator increases at same extent and suddenly decreases up to time t* and thereafter remains same throughout as shown in the Figure 1.

Situation 2: This is a case where both the species Prey and Predator decrease initially. The Predator is the dominant one as it dominates the prey throughout. In course of time we notice a steady variation with no appreciable growth rate in both the species. Further, the immigration coefficient is less than migration coefficient. (Figure 2).

Situation 3: Initially the prey is dominant over the predator but if suffers a steep fall and after an instant t* the dominance is reversed. The predator has a steep rise initially and then suffers fall. The migration coefficient is same as the interaction coefficient a_{12} and the natural growth is equal to a_{11}. (Figure 3).

Situation 4: Initially there is a steep rise in the growth rate of the predator as it dominates the prey and after a time instant t* there is a fall in the growth rate of the predator and the prey outnumbers the predator. Also we notice that the prey has a steady rise as illustrated in Figure 4.

Situation 5: In this case the predator would always dominate over the prey. The interaction coefficient a_{12} is same as the migration coefficient. Further we notice that both the species have a steady variation with no appreciable growth rates. (Figure 5).

Situation 6: Initially the predator is dominant over the prey up to a time instant t* after which the prey dominates over the predator. In this case we observe that the prey initially has a steady increase and in course of time both the species coexist with a steady variation and no appreciable growth. (Figure 6).

Situation 7: In this case the initial conditions of the prey and predator species are identical. Initially there is a steep rise in the growth rate of the predator as it dominates the prey and after a time instant t* there is a fall in the growth rate of the predator and the prey outnumbers the predator. (Figure 7).

Situation 8: In this case both the species suffer a fall initially and after an instant t* the prey exists at a very low rate and the Predator becomes extinct at t*. The natural growth rate of prey is very low. (Figure 8).

The present paper deals with an investigation on the stability of two species ecology consisting of a prey-predator with limited resources. The system comprises of a prey (S_1), a predator (S_2) that survives upon S_1 with migration of prey and immigration of predator. All possible equilibrium points of the model are identified and criteria for their stability is discussed. It is observed that, in all four equilibrium points,

(i) The prey washed out state is stable when $a_1 < b_1$ and is neutrally stable when $a_1 = b_1$.

(ii) The predator washed out state is stable when $M < a_1;\ a_2 + I < d$ and is neutrally stable when $M = a_1, a_2 + I = d; M = a_1, a_2 + I < d$ and $M < a_1, a_2 + I = d$

(iii) The normal steady state is stable when λ_1, λ_2 are negative.

Further, the trajectories of solution curves for all equilibrium states are illustrated.

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