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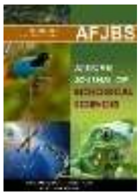

FERMATEAN FUZZY SET ACTING ON FUZZY TOPOLOGICAL STRUCTURES

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Abstract: In this paper, we study the concept of Fermatean fuzzy topological properties of such algebras such as connectedness, strong connectedness and compact Haussdorff space. We also obtain the characteristic of the homomorphic image and inverse image of Fermatean fuzzy topological BCC-ideals (BCK-ideals) of BCC-algebras (BCK-algebras).

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1.Introduction: In classical set theory, the membership of elements in a set is assessed in binary terms according to a bivalent condition: an element either belongs or does not belong to the set. As an extension, fuzzy set theory (See [22]) permits the gradual assessment of the membership of elements in a set; this is described with the aid of a membership function valued in the real unit interval [0,1]. As a generalization of fuzzy set, Atanassov [1] created intuitionistic fuzzy set. Intuitionistic fuzzy set is widely used in all fields (See [4, 5, 12, 18] for applications in algebraic structures). In 2013, Yager [19, 20, 21] introduced Pythagorean fuzzy set and compared it with intuitionistic fuzzy set. Pythagorean fuzzy set is a new extension of intuitionistic fuzzy set that conducts to simulate the vagueness originated by the real case that might arise in the sum of membership and non-membership is bigger than 1. Pythagorean fuzzy set is applied to groups (See [2]), UP-algebras (See [15]) and topological spaces (See [14]). Senapati et.al [16] introduced Fermatean fuzzy set which is another extension of intuitionistic fuzzy sets and it is applied to groups (See [17]). Ibrahim et.al [9] introduced Fermatean fuzzy sets and applied it to topological spaces. In this paper, we study the concept of Fermatean fuzzy topological properties of such algebras such as connectedness, strong connectedness and compact Haussdorff space. We also obtain the characteristic of the homomorphic image and inverse image of Fermatean fuzzy topological BCC-ideals (BCK-ideals) of BCC-algebras (BCK-algebras).

2. Preliminaries of BCC-algebras(BCK-algebras)

In this section, we first review some definitions and properties which will be used in the sequel.

A non-empty set G with a constant 0 and binary operation $*$ is called a BCC-algebra if it satisfies the following conditions:

a) $\Big(\big((x * y) * (z * y)\big) * (x * z) = 0\Big)$

b) $x * x = 0$

c) $0 * x = 0$

d) $x * 0 = 0$

e) $x * y = 0\,, y * x = 0 \;\Rightarrow x = y$

for all $x, y, z \in G$. In BCC-algebra, the following equality holds $(x * y) * x = 0$.

Obviously, any BCK-algebra is BCC-algebra but there exist BCC-algebras which are not necessarily BCK-algebra. We note that a BCC-algebra is BCK-algebra if and only if it satisfies the equality $(x * y) * z = (x * z) * y$.

A non-empty subset ‘S’ of a BCK-algebra ‘G’ is called a sub algebra of G if it is closed under the BCC-operation. Such algebra contains the constant 0 and it is clearly a BCC-algebra, but some sub algebras may be also BCK-algebras. Moreover, there exit BCC-algebras in which all sub algebras are BCK-algebras.

A mapping $\varphi: G_1 \to G_2$ of BCC-algebras is called a homomorphism if

$\varphi(x * y) = \varphi(x) * \varphi(y)$ holds, for all $x, y \in G_1$.

For a non-empty given set G, let I be the closed unit interval [0, 1]. Then, an Fermatean fuzzy set is an object of the form $A = \{\langle x, {\delta_A}^3(x), {\lambda_A}^3(x)\rangle / x \in G\}$, when the mappings ${\delta_A}^3: G \to I$ and ${\lambda_A}^3: G \to I$ denote the degree of membership (namely, $\delta_A(x)$) and the degree of non-membership (namely, $\lambda_A(x)$) of each element $x \in G$ to the object 'A' respectively satisfying $0 \leq {\delta_A}^3(x) + {\lambda_A}^3(x) \leq 1$ for all $x \in G$.

The complement of the Fermatean fuzzy set A is $A^C = \{\langle x, {\lambda_A}^3(x), {\delta_A}^3(x)\rangle / x \in G\}$. Obviously, every fuzzy set A on a non-empty G is an (3,3)-fuzzy set of the form $A = \{\langle x, {\delta_A}^3(x), 1 - {\lambda_A}^3(x)\rangle / x \in G\}$. For the sake of simplicity, we just write $A = \langle {\delta_A}^3, {\lambda_A}^3\rangle$ instead of $A = \{\langle x, \delta_A(x), \lambda_A(x)\rangle / x \in G\}$.

The Fermatean fuzzy sets $0\sim$ and $1\sim$ in G are defined by $0\sim = \{\langle x, 0, 1\rangle : x \in G\}$ and $1\sim = \{\langle x, 1, 0\rangle : x \in G\}$, respectively.

If φ is a mapping which maps a set G_1 into another set G_2, then the following statements hold:

(a) If $B = \{\langle y, {\delta_B}^3(y), {\lambda_B}^3(y)\rangle / y \in G_2\}$ is a Fermatean fuzzy set in G_2, then the pre image of B under φ, denoted by $\varphi^{-1}(B)$, is still a Fermatean fuzzy set in G_1, we now write
$\varphi^{-1}(B) = \{\langle x,\ \varphi^{-1}(\delta_B)(x),\ \varphi^{-1}(\lambda_B)(x)\rangle / x \in G_1\}$.

(b) If $A = \{\langle x, {\delta_A}^3(x), {\lambda_A}^3(x)\rangle / x \in G_1\}$ is a Fermatean fuzzy set in G_1, then the image of A under φ, denoted by $\varphi(A)$, is also a Fermatean fuzzy set in G_2, which is defined by $\varphi(A) = \{\langle y, \varphi_{sup}(\delta_A)(y), \varphi_{inf}(\lambda_A)(y)\rangle : y \in G_2\}$, where

$$\varphi_{sup}(\delta_A)(y) = \begin{cases} \sup\limits_{x \in \varphi^{-1}(y)} \delta_A(x), & \text{if } \varphi^{-1}(y) \neq \phi \\ 0, & elsewhere \end{cases}$$

$$\varphi_{inf}(\lambda_A)(y) = \begin{cases} \inf\limits_{x \in \varphi^{-1}(y)} \lambda_A(x), & \text{if } \varphi^{-1}(y) \neq \phi \\ 1, & otherwise \end{cases}$$

for each $y \in G_2$.

Proposition-2.1: Let A, $A_i (i \in I)$ be Fermatean fuzzy set in G_1 and B a Fermatean fuzzy set in G_2. If $\varphi: G_1 \to G_2$ is a function, then the following properties hold for the function φ:

(a) If φ is surjective, then $\varphi(\varphi^{-1}(B)) = B$

(b) $\varphi^{-1}(\cup_{i=1}^{n} A_i) = \cup_{i=1}^{n} \varphi^{-1}(A_i)$

(c) $\varphi^{-1}(1\sim) = 1\sim$

(d) $\varphi^{-1}(0\sim) = 0\sim$

(e) $\varphi(1\sim) = 1\sim$, if φ is surjective

(f) $\varphi(0\sim) = 0\sim$

Definition-2.2: A Fermatean fuzzy topology on a non-empty set G is a family τ of Fermatean fuzzy sets in G which satisfies the following conditions:

(i) $0\sim, 1\sim \in \tau$

(ii) If $G_1, G_2 \in \tau$, then $G_1 \cap G_2$

(iii) If $G_j \in \tau$ for all $j \in J$, then $\bigcup_{i\in I} G_i \in \tau$

The pair (G, τ) is called a Fermatean fuzzy topological space and any Fermatean fuzzy set in τ is called a Fermatean fuzzy open sets in G. The topology τ on a Fermatean fuzzy topological space is said to be an indiscrete Fermatean fuzzy topology if its only elements are $0\sim$ and $1\sim$. On the other hand, Fermatean fuzzy topology τ on a space G is said to be discrete Fermatean fuzzy topology if the topology Fermatean fuzzy topology τ contains all Fermatean fuzzy subsets of G.

If A is a Fermatean fuzzy set in a Fermatean fuzzy topological space (G, τ), then the induced Fermatean fuzzy topological space on A is the family of Fermatean fuzzy sets in A which are the intersection with A of Fermatean fuzzy sets in G. The induced Fermatean fuzzy topology is denoted by τ_A, and the pair (A, τ_A) is called an fuzzy subspace of (G, τ).

Let (G_1, τ_1) and (G_2, τ_2) be two Fermatean fuzzy topological spaces and $\varphi: (G_1, \tau_1) \to (G_2, \tau_2)$ a function. Then φ is said to be Fermatean fuzzy continuous function if and only if the pre image of each Fermatean fuzzy set in τ_2 is a Fermatean fuzzy set in τ_1. Let (G_1, τ_1) and (G_2, τ_2) be two Fermatean fuzzy topological spaces and $\varphi: (G_1, \tau_1) \to (G_2, \tau_2)$ a function. Then φ is said to be Fermatean fuzzy open if and only if the image of each Fermatean fuzzy set in τ_1 is a Fermatean fuzzy set in τ_2.

3. Fermatean fuzzy topological sub algebras

Definition-3.1: Fermatean fuzzy set $A = \langle {\delta_A}^3, {\lambda_A}^3 \rangle$ in G is called Fermatean fuzzy sub algebra of G if it satisfies the following conditions;

(3, 3) FS_1 : ${\delta_A}^3(x * y) \geq \min\{{\delta_A}^3(x), {\delta_A}^3(y)\}$

(3, 3) FS_2 : ${\lambda_A}^3(x * y) \leq \max\{{\lambda_A}^3(x), {\lambda_A}^3(y)\}$, for all $x, y \in G$.

Example-3.2: Let $G = \{0, 1, 2, 3, 4\}$ be a BCC-algebra with the following Cayley table.

+	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0

A.Sheela Roselin /Afr.J.Bio.Sc. 6(13)(2024).3460-3472

2	2	2	0	0	0
3	3	3	1	0	0
4	4	3	4	3	0

Let $A = \langle {\delta_A}^3, {\lambda_A}^3 \rangle$ be a Fermatean fuzzy set in G defined by ${\delta_A}^3(4) = 0.07$, ${\delta_A}^3(x) = 0.6$, ${\lambda_A}^3(x) = 0.5$ and ${\lambda_A}^3(4) = 0.06$ for all $x \neq 4$. Then A is a Fermatean fuzzy sub algebra of G.

Definition-3.3: Let τ_1 and τ_2 be Fermatean fuzzy topologies on BCC-algebras G_1 and G_2 respectively. A function $\varphi: (G_1, \tau_1) \to (G_2, \tau_2)$ is called a Fermatean fuzzy continuous function which maps (G_1, τ_1) and (G_2, τ_2) if φ satisfies the following conditions:

(i) For every $A \in \tau_2$, $\varphi^{-1}(A) \in \tau_1$.

(ii) For every Fermatean fuzzy sub algebra A (of G_2) in τ_2, $\varphi^{-1}(A)$ is a Fermatean fuzzy sub algebra (of G_1) in τ_1.

Proposition-3.4: If τ_1 is a Fermatean fuzzy topology on a BCC-algebra G_1 and τ_2 is a Fermatean fuzzy topology on a BCC-algebra G_2, then every function $\varphi: (G_1, \tau_1) \to (G_2, \tau_2)$ is a (3, 3)-fuzzy continuous function.

Proof: Since τ_2 is an indiscrete Fermatean fuzzy topology, $\tau_2 = (0\sim, 1\sim)$.

Let $\varphi: G_1 \to G_2$ be any mapping of BCC-algebras. Then, every member of τ_2 is a Fermatean fuzzy topology on a BCC-algebra G_2.

We now show that φ is a Fermatean fuzzy continuous function. We only need to prove that for every $A \in \tau_2$, $\varphi^{-1}(A) \in \tau_1$.

For this purpose, we let $0\sim \in \tau_2$. Then for any $x \in G_1$, we have

$\varphi^{-1}(0\sim)(x) = 0\sim(\varphi(x)) = 0 = 0\sim(x)$. This show that $\left(\varphi^{-1}(0\sim)\right) = 0\sim \in \tau_1$.

On the other hand, if $1\sim \in \tau_2$ and $x \in G_1$, then

$\varphi^{-1}(1\sim)(x) = 1\sim(\varphi(x)) = 1 = 1\sim(x)$. Thus $\left(\varphi^{-1}(1\sim)\right) = 1\sim \in \tau_1$.

This show that φ is indeed a Fermatean fuzzy continuous function of G_1 to G_2.

Theorem-3.5: Let τ_1 and $\tau_2$be any two discrete Fermatean fuzzy topologies defined on the BCC-algebras G_1 and G_2 respectively. Then every homomorphism $\varphi: (G_1, \tau_1) \to (G_2, \tau_2)$ is a Fermatean fuzzy continuous function.

Proof: Since τ_1 and τ_2 are discrete Fermatean fuzzy topologies on the BCC-algebras G_1 and G_2 respectively, we have $\varphi^{-1}(A) \in \tau_1$ for every $A \in \tau_2$.

We note that φ is not the usual inverse homomorphism from G_2 to G_1.

Let $A = \langle {\delta_A}^3, {\lambda_A}^3 \rangle$ be a Fermatean fuzzy sub algebra (of G_2) in τ_2. Then for $x, y \in G_1$, we have $\left(\varphi^{-1}\left({\delta_A}^3\right)\right)(x * y) = {\delta_A}^3\left(\varphi(x * y)\right)$

$$= {\delta_A}^3(\varphi(x) * \varphi(y))$$
$$\geq \min\{{\delta_A}^3(\varphi(x)), {\delta_A}^3(\varphi(y))\}$$
$$= \min\left\{\left(\varphi^{-1}({\delta_A}^3)\right)(x), \left(\varphi^{-1}({\delta_A}^3)\right)(y)\right\} \text{ and}$$

$$\left(\varphi^{-1}({\lambda_A}^3)\right)(x * y) = {\lambda_A}^3(\varphi(x * y))$$
$$= {\lambda_A}^3(\varphi(x) * \varphi(y))$$
$$\leq \max\{{\lambda_A}^3(\varphi(x)), {\lambda_A}^3(\varphi(y))\}$$
$$= \max\left\{\left(\varphi^{-1}({\lambda_A}^3)\right)(x), \left(\varphi^{-1}({\lambda_A}^3)\right)(y)\right\}$$

Hence $\varphi^{-1}(A)$ is a Fermatean fuzzy sub algebra (of G_1) in τ_1 and consequently, φ is a Fermatean fuzzy continuous function which maps (G_1, τ_1) to (G_2, τ_2).

Definition-3.6: Let (G_1, τ_1) and (G_2, τ_2) be Fermatean fuzzy topology sub algebras. A function $\varphi: (G_1, \tau_1) \to (G_2, \tau_2)$ is said to be a Fermatean fuzzy homomorphism if it satisfies the following conditions:

- φ is an injective and surjective function.
- φ is fuzzy continues function which maps G_1 to G_2.
- φ^{-1} is fuzzy continues function which maps G_2 to G_1.

Definition-3.7: Let τ be a Fermatean fuzzy topology on a BCC-algebra G. A Fermatean fuzzy topology (G, τ) is a Fermatean fuzzy Hausdorff space if and only if for any distinct Fermatean fuzzy points $x_1, x_2 \in G$, there exit (3, 3)- fuzzy topologies $F_1 = \langle \delta^3_{F_1}, \lambda^3_{F_1} \rangle$ and $F_2 = \langle \delta^3_{F_2}, \lambda^3_{F_2} \rangle$ such that $\delta^3_{F_1}(x_1) = 1$, $\lambda^3_{F_1}(x_1) = 0$, $\delta^3_{F_2}(x_2) = 1$, $\lambda^3_{F_2}(x_2) = 0$ and $F_1 \cap F_2 = 0\sim$.

Theorem-3.8: Let τ_1 and τ_2 be Fermatean fuzzy topologies on BCC-algebras G_1 and G_2 respectively and let $\varphi: (G_1, \tau_1) \to (G_2, \tau_2)$ be a Fermatean fuzzy homomorphism. Then G_1 is a Fermatean fuzzy Hausdorff space if and only if G_2 is a Fermatean fuzzy Hausdorff space.

Proof: Suppose that G_1 is a Fermatean fuzzy Hausdorff space.

Let x_1, x_2 be the Fermatean fuzzy points in τ_2 with $x \neq y$ where $x, y \in G_1$.

Then $\varphi^{-1}(x) \neq \varphi^{-1}(y)$ because φ is injective function.

For $z \in G_1$, $\left(\varphi^{-1}(x_1)\right)(z) = x_1\left(\varphi(z)\right)$

$$= \begin{cases} s \in [0,1], & \text{if } \varphi(z) = x \\ 0, & \text{if } \varphi(z) \neq x \end{cases} = \begin{cases} s \in [0,1], & \text{if } z = \varphi^{-1}(x) \\ 0, & \text{if } z \neq \varphi^{-1}(x) \end{cases}$$
$$= \left(\varphi^{-1}(x)\right)_1(z).$$

That is, $\left(\varphi^{-1}(x_1)\right)(z) = \left(\varphi^{-1}(x)\right)_1(z)$ for all $z \in G$. Hence $\varphi^{-1}(x_1)$=$\left(\varphi^{-1}(x)\right)_1$.

Similarly we can also prove that $\varphi^{-1}(x_2)=\left(\varphi^{-1}(x)\right)_2$. Now by the definition of a Fermatean fuzzy Hausdorff space, there exist Fermatean fuzzy orders F_1 and F_2 of $\varphi^{-1}(x_1)$ and $\varphi^{-1}(x_2)$ respectively such that $F_1 \cap F_2 = 0\sim$. Since φ is an (3, 3)- fuzzy continuous map from G_2 to G_1, there exist Fermatean fuzzy orders $\varphi(F_1)$ and $\varphi(F_2)$ of x_1 and x_2 respectively such that $\varphi(F_1) \cap \varphi(F_2) = \varphi(F_1 \cap F_2) = \varphi(0\sim) = 0\sim$. This implies that G_2 is a Fermatean fuzzy Hausdorff space.

Conversely, if (G_2, τ_2) is a Fermatean fuzzy Hausdorff space, then by using a similar argument as above and by the fact that both φ and φ^{-1} are Fermatean fuzzy continuous functions. We can easily prove that (G_1, τ_1) is a Fermatean fuzzy Hausdorff space. Hence the proof.

Definition-3.9: Let τ be a Fermatean fuzzy topology on a BCC-algebra G. Then (G, τ) is called a Fermatean fuzzy C_5-disconnected space if there exists a Fermatean fuzzy open and closed set F such that $F \neq 0\sim$ and $F \neq 1\sim$.

Theorem-3.10: Let τ_1 and τ_2 be Fermatean fuzzy topologies on BCC-algebras G_1 and G_2 respectively and let $\varphi: G_1 \rightarrow G_2$ be a Fermatean fuzzy continuous surjective function. If (G_1, τ_1) is a Fermatean fuzzy C_5-connected space, then (G_2, τ_2) is also a Fermatean fuzzy C_5-connected space.

Proof: Assume that (G_2, τ_2) is a Fermatean fuzzy C_5-disconnected. Then there exists a Fermatean fuzzy open and closed set F such that $F \neq 0\sim$ and $F \neq 1\sim$.

Since φ is an (3, 3)- fuzzy continuous function, $\varphi^{-1}(F)$ is both Fermatean fuzzy open set and Fermatean fuzzy closed set. In this case $\varphi^{-1}(F) \neq 0\sim$ or $\varphi^{-1}(F) \neq 1\sim$.

Since, $F = \varphi\left(\varphi^{-1}(F)\right) = \varphi(0\sim) = 0\sim$ and $F = \varphi\left(\varphi^{-1}(F)\right) = \varphi(1\sim) = 1\sim$,

We see that these results contradict to our assumption.

Hence the space (G_2, τ_2) must be a Fermatean fuzzy C_5-connected space.

Definition-3.11: Let τ be a Fermatean fuzzy topology on a BCC-algebra G. A Fermatean fuzzy topology set (G, τ) is called a Fermatean fuzzy disconnected space if there exist Fermatean fuzzy open sets $A \neq 0\sim$ and $B \neq 0\sim$ such that $A \cup B = 0\sim$. Naturally, we call the set (G, τ) Fermatean fuzzy connected if (G, τ) is not Fermatean fuzzy disconnected.

Theorem-3.12: Let τ_1 and τ_2 be Fermatean fuzzy topologies on BCC-algebras G_1 and G_2 respectively and let $\varphi: (G_1, \tau_1) \rightarrow (G_2, \tau_2)$ be a Fermatean fuzzy continuous and surjective mapping. If G_1 is a Fermatean fuzzy connected space, then so is G_2.

Proof: Suppose that G_2 is a Fermatean fuzzy disconnected, then there exist Fermatean fuzzy open sets $C \neq 0\sim$ and $D \neq 0\sim$ in G_2 such that $C \cup D = 1\sim$ and $C \cap D = 0\sim$.

Since φ is a Fermatean fuzzy continuous function, $A = \varphi^{-1}(C)$ and $B = \varphi^{-1}(D)$ are Fermatean fuzzy open sets in G_1.

Clearly, $C \neq 0\sim$ implies that $A = \varphi^{-1}(C) \neq 0\sim$ and $D \neq 0\sim$ implies that $B = \varphi^{-1}(D) \neq 0\sim$.

Now $C \cup D = 1\sim$.

$\Rightarrow \varphi^{-1}(C \cup D) = \varphi^{-1}(1\sim)$.

$\Rightarrow \varphi^{-1}(C) \cup \varphi^{-1}(D) = 1\sim$ implies $A \cup B = 1\sim$ and

$C \cap D = 0\sim \Rightarrow \varphi^{-1}(C \cap D) = \varphi^{-1}(0\sim)$

$\Rightarrow \varphi^{-1}(C) \cap \varphi^{-1}(D) = 0\sim$ implies $A \cap B = 0\sim$.

This clearly contradicts our hypothesis.

Hence G_2 is a Fermatean fuzzy connected space.

Definition-3.13: A Fermatean fuzzy topology space (G, τ) is said to be a Fermatean fuzzy strongly connected, if there exist no non-zero Fermatean fuzzy closed sets A and B in G such that $\delta_A^3 + \delta_B^3 \leq 1$ and $\lambda_A^3 + \lambda_B^3 \geq 1$.

The following fact follows immediately from above definition.

Propositon-3.14: G is a Fermatean fuzzy strongly connected if and only if there exist Fermatean fuzzy open sets A and B in G such that $A \neq 1\sim \neq B$ and $\delta_A^3 + \delta_B^3 \geq 1$, $\lambda_A^3 + \lambda_B^3 \leq 1$.

We now formulate the following theorem.

Theorem-3.15: Let τ_1 and τ_2 be Fermatean fuzzy topologies on BCC-algebras G_1 and G_2 respectively and let $\varphi: (G_1, \tau_1) \to (G_2, \tau_2)$ be a Fermatean fuzzy continuous and surjective mapping. If G_1 is a Fermatean fuzzy strongly connected, then so is G_2.

Proof: Suppose that G_2 is not a Fermatean fuzzy strongly connected. Then there exist Fermatean fuzzy open sets C and D in G_2 with $C \neq 0\sim$ and $D \neq 0\sim$ so that $\delta_C^3 + \delta_D^3 \leq 1$ and $\lambda_C^2 + \lambda_D^2 \geq 1$. Since φ is a Fermatean fuzzy continuous function, $\varphi^{-1}(C)$ and $\varphi^{-1}(D)$ are $(3, 3)$-fuzzy closed sets in G_1. Now we can deduce the following equalities;

$$\delta^3_{\varphi^{-1}(C)} + \delta^3_{\varphi^{-1}(D)} = \varphi^{-1}(\delta_C^3) + \varphi^{-1}(\delta_D^3)$$

$$= \delta_C^3 \text{ o } \varphi + \delta_D^3 \text{ o } \varphi \leq 1 \text{ (Since } \delta_C^3 + \delta_D^3 \leq 1\text{)},$$

$$\lambda^3_{\varphi^{-1}(C)} + \lambda^3_{\varphi^{-1}(D)} = \varphi^{-1}(\lambda_C^3) + \varphi^{-1}(\lambda_D^3)$$

$$= \lambda_C^3 \text{ o } \varphi + \lambda_D^3 \text{ o } \varphi \geq 1 \text{ (Since } \lambda_C^3 + \lambda_D^3 \geq 1\text{)}.$$

$\varphi^{-1}(C) \neq 0\sim$ and $\varphi^{-1}(D) \neq 0\sim$. This contradicts our hypothesis. Hence G_2 is a Fermatean fuzzy strongly connected space.

Definition-3.16: Let τ be a Fermatean fuzzy topology on a BCC-algebra G and A be a Fermatean fuzzy BCC-algebra with Fermatean fuzzy topology τ_A. Then A is called a Fermatean fuzzy topological BCC-sub algebra if the self-mapping $\gamma_a: (A, \tau_A) \to (A, \tau_A)$ defined by $\gamma_a(x) = x * a$ for all $a \in G$, is a Relatively Fermatean fuzzy continuous function.

Theorem-3.17: Let $\varphi: G_1 \to G_2$ be a homomorphism of BCC-algebras and let τ and τ^* be Fermatean fuzzy topologies on G_1 and G_2 respectively such that $\tau = \varphi^{-1}(\tau^*)$. If B is a Fermatean fuzzy topological BCC-sub algebra in G_2, then $\varphi^{-1}(B)$ is a Fermatean fuzzy topological BCC-sub algebra in G_1.

Theorem-3.18: Let $\varphi: G_1 \to G_2$ be an isomorphism of BCC-algebras. Let τ and τ^* be the respectively Fermatean fuzzy topologies on the spaces G_1 and G_2 such that $\tau = \varphi^{-1}(\tau^*)$. If A is a Fermatean fuzzy topological BCC-sub algebra in G_1, then $\varphi^{-1}(A)$ is a Fermatean fuzzy topological BCC-sub algebra in G_2.

4. Fermatean fuzzy topological BCC-ideals

Definition-4.1: Fermatean fuzzy set $A = \{\langle \delta_A, \lambda_A \rangle\}$ in a BCK-algebra G is called a Fermatean fuzzy BCK-ideal of G if the following conditions are satisfied;

(i) $\delta_A^3(0) \geq \delta_A^3(x)$ and $\lambda_A^3(0) \leq \lambda_A^3(x)$,

(ii) $\delta_A^3(x) \geq \min\{\delta_A^3(x * y), \delta_A^3(y)\}$

(iii) $\lambda_A^3(x) \leq \max\{\lambda_A^3(x * y), \lambda_A^3(y)\}$ for all $x, y \in G$.

Definition-4.2: An (3, 3)-fuzzy set $A = \langle \delta_A, \lambda_A \rangle$ in G is called a Fermatean fuzzy BCC-ideal of G if it satisfies the following conditions;

(3, 3) F_1: $\delta_A^3(0) \geq \delta_A^3(x)$ and $\lambda_A^3(0) \leq \lambda_A^3(x)$

(3, 3) F_2: $\delta_A^3(x * z) \geq \min\{\delta_A^3((x * y) * z), \delta_A^3(y)\}$

(3, 3) F_3: $\lambda_A^3(x * z) \leq \max\{\lambda_A^3((x * y) * z), \lambda_A^3(y)\}$ for all $x, y, z \in G$.

Putting $z = 0$ in Fermatean F_2 and Fermatean F_3, then we can easily see that a Fermatean fuzzy BCC-ideal is a Fermatean fuzzy BCK-ideal. However, the converse does not hold.

Example-4.3: Let $G = \{0, 1, 2, 3, 4, 5\}$ be a BCC-algebra with the following Cayley table;

+	0	1	2	3	4	5
0	0	0	0	0	0	0
1	1	0	0	0	0	1
2	2	2	0	0	1	1
3	3	2	1	0	1	1
4	4	4	4	4	0	1
5	5	5	5	5	5	0

Let $A = \langle \delta_A, \lambda_A \rangle$ be a Fermatean fuzzy set in G defined by $\delta_A^3(5) = 0.02$, $\delta_A^3(x) = 0.4$,

$\lambda_A^3(5) = 0.2$ and $\lambda_A^3(x) = 0.04$ for all $x \neq 5$. Then A is a Fermatean fuzzy BCC-ideal of a BCC-algebra G.

Theorem-4.4: Let φ be a homomorphism of a BCC-algebra G_1 into a BCC-algebra G_2 and B be a Fermatean fuzzy BCC-ideal of G_2. Then $\varphi^{-1}(B)$ is a Fermatean fuzzy BCC-ideal of G_1.

Proof: It can be easily seen that

$\delta^3_{\varphi^{-1}(B)}(0) \geq \delta^3_{\varphi^{-1}(B)}(x)$ and $\lambda^3_{\varphi^{-1}(B)}(0) \leq \lambda^3_{\varphi^{-1}(B)}(x)$, for all $x \in G_1$.

For any $x, y, z \in G_1$, we can deduce the following

$$\begin{aligned}\delta^3_{\varphi^{-1}(B)}(x * z) &= \delta^3_B(\varphi(x * z)) \\ &\geq \min\left\{\delta^3_B\left(\varphi((x * y) * z)\right), \delta^3_B(\varphi(y))\right\} \\ &= \min\left\{\delta^3_B\left((\varphi(x) * \varphi(y)) * \varphi(z)\right), \delta^3_B(\varphi(y))\right\} \\ &= \min\left\{\delta^3_{\varphi^{-1}(B)}((x * y) * z), \delta^3_{\varphi^{-1}(B)}(y)\right\}.\end{aligned}$$

Also

$$\begin{aligned}\lambda^3_{\varphi^{-1}(B)}(x * z) &= \lambda^3_B(\varphi(x * z)) \\ &\leq \max\left\{\lambda^3_B\left(\varphi((x * y) * z)\right), \lambda^3_B(\varphi(y))\right\} \\ &= \max\left\{\lambda^3_B\left((\varphi(x) * \varphi(y)) * \varphi(z)\right), \lambda^3_B(\varphi(y))\right\} \\ &= \max\left\{\lambda^3_{\varphi^{-1}(B)}((x * y) * z), \lambda^3_{\varphi^{-1}(B)}(y)\right\}\end{aligned}$$

Hence $\varphi^{-1}(B)$ is a Fermatean fuzzy BCC-ideal of G_1.

Corollarly-4.5: Let φ be a homomorphism of a BCC-algebra G_1 into a BCC-algebra G_2 and let B be a Fermatean fuzzy BCK-ideal of G_2. Then $\varphi^{-1}(B)$ is a Fermatean fuzzy BCK-ideal of G_1.

Since an $(3, 3)$- fuzzy BCC-ideal / BCK-ideal is a Fermatean fuzzy sub algebra, as a consequence of the above results and theorem-3.17, we obtain the following corollary:

Corollarly-4.6: Let $\varphi: (G_1, \tau_1) \to (G_2, \tau_2)$ be a homomorphism of the BCC-algebras. Let τ_1 and τ_2 be the Fermatean fuzzy topologies on G_1 and G_2 respectively such that $\tau_2 = \varphi^{-1}(\tau_1)$. If B is a Fermatean fuzzy topological BCC-ideal / BCK-ideal of G_2 with the membership function δ^3_B, then $\varphi^{-1}(B)$ is a Fermatean fuzzy topological BCC-ideal / BCK-ideal of G_1 with the membership function $\delta^3_{\varphi^{-1}(B)}$.

Theorem-4.7: Let φ be a homomorphism of a BCC-algebra G_1 into a BCC-algebra G_2. If A is a Fermatean fuzzy BCC-ideal of G_1, then the homomorphic image $\varphi(A)$ of A is still a Fermatean fuzzy BCC-ideal of G_2.

Proof: Let A be a Fermatean fuzzy topological BCC-ideal of G_1. Then, it is trivial that $\delta^3_{\varphi(A)}(0) \geq \delta^3_{\varphi(A)}(x)$ and $\lambda^3_{\varphi(A)}(0) \leq \lambda^3_{\varphi(A)}(x)$, for all $x \in G_2$.

Take $x, y, z \in G_2$ and let $x_0 \in \varphi^{-1}(x)$, $y_0 \in \varphi^{-1}(y)$, $z_0 \in \varphi^{-1}(z)$ such that $\delta^3_A(x_0) = \sup_{t\in \varphi^{-1}(x)} t$, $\delta^3_A(y_0) = \sup_{t\in \varphi^{-1}(y)} t$ and $\delta^3_A(z_0) = \sup_{t\in \varphi^{-1}(z)} t$.

Then we can deduce the following,

$$\begin{aligned}
\delta^3_{\varphi(A)}(x * z) &= \sup_{t\in \varphi^{-1}(x*z)} \left(\delta^3_A(t)\right) \\
&\geq \delta^3_A(x_0 * z_0) \\
&\geq \min\{\delta^3_A\big((x_0 * y_0) * z_0\big), \delta^3_A(y_0)\} \\
&= \min\left\{\sup_{t\in \varphi^{-1}((x*y)*z)} \left(\delta^3_A(t)\right), \sup_{t\in \varphi^{-1}(y)} \left(\delta^3_A(t)\right)\right\} \\
&= \min\{\delta^3_{\varphi(A)}\big((x * y) * z\big), \delta^3_{\varphi(A)}(y)\}
\end{aligned}$$

and $$\begin{aligned}
\lambda^3_{\varphi(A)}(x * z) &= \inf_{t\in \varphi^{-1}(x*z)} \left(\lambda^3_A(t)\right) \leq \lambda^3_A(x_0 * z_0) \\
&\leq \max\{\lambda^3_A\big((x_0 * y_0) * z_0\big), \lambda^3_A(y_0)\} \\
&= \max\left\{\inf_{t\in \varphi^{-1}((x*y)*z)} \left(\lambda^3_A(t)\right), \inf_{t\in \varphi^{-1}(y)} \left(\lambda^3_A(t)\right)\right\} \\
&= \max\{\lambda^3_{\varphi(A)}\big((x * y) * z\big), \lambda^3_{\varphi(A)}(y)\}
\end{aligned}$$

Hence $\varphi(A) = \langle \varphi_{\sup}(\delta_A), \varphi_{\inf}(\lambda_A)\rangle$ is induced a Fermatean fuzzy BCC-ideal of G_2.

Putting $z = 0$ in the above theorem, we obtain:

Corollarly-4.8: Let φ be a homomorphism of a BCC-algebra G_1 into a BCC-algebra G_2. If A is a Fermatean fuzzy BCK-ideal of G_1, then the homomorphic image $\varphi(A)$ of A is still a Fermatean fuzzy BCK-ideal of G_2.

Summing up theorem-3.18, theorem-4.7 and corollary-4.8, we conclude the following theorem.

Theorem-4.9: Let $\varphi: G_1 \to G_2$ be an isomorphism of BCC-algebras. Let τ and τ^* be the respectively Fermatean fuzzy topologies on the spaces G_1 and G_2 such that $\varphi(\tau) = \tau^*$. If A is a Fermatean fuzzy topological BCC-ideal / BCK-ideal in G_1, then $\varphi(A)$ is also a Fermatean fuzzy topological BCC-ideal / BCK-ideal in G_2.

Conclusion: Here we studied the concept of Fermatean fuzzy topological properties of such algebras such as connectedness, strong connectedness and compact Haussdorff space. We also discussed the characteristic of the homomorphic image and inverse image of Fermatean fuzzy topological BCC-ideals (BCK-ideals) of BCC-algebras (BCK-algebras).

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