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NON COMMON FIXED POINT OF NON-EXPANSIVE MAPPING FOR MULTISTEP ITERATION IN BANACH SPACES

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Abstract

In the current paper we discussed some applications of f non-expansive and quasi-non-expansive mapping in Banach space and matric space along with strong and weak convergence of the sequence of certain iterates to a fixed point of quasi-non-expansive map. we show that $\{X\_n\}$ converges weakly to a common fixed point of T and I the sequence $\{X_n\}$ contain a subsequence which converges weakly to a point in K Let $\{X_{nk}\}$ and $\{X_{mk}\}$be two subsequences of $\{X_n\}$ which converges weakly to f and q, respectively. We will show that f=g. Suppose that E satisfies Opial's condition ans that f ≠ q is in weak limit set of the sequence $\{X_n\}$. Then $\{X_{nk}\} \rightarrow$ and $\{X_{mk}\} \rightarrow$ q, respectively since $\lim_{n\to\infty}$ II X_n-f II exists for any f $\in$ F (T) $\cap$ F (I) by Opial's condition.

Keywords: Non Common Fixed Point, Non-Expansive Mapping, Banach Spaces, Multistep Iteration

1. INTRODUCTION:

There are many results on fixed point on non-expansive and quasi-non-expansive mapping in Banach space and matric space for example, the strong and weak convergence of the sequence of certain iterates to a fixed point of quasi-non-expansive map was studies by Petryshyn and Williamson [1], Aoyama and Kohsaka [17], Kirk and Sims [18], Temir and Gul [19], and Zhou et al. [20] Their analysis was related to the convergence of Mann iterates studies by Doston [2]. Subsequently the convergence of Ishikawa iterates of quasi-non-expansive mapping in Banach space was discussed by Ghosh and Debnath [3]. In the weakly convergence theorem for i-asymptotically quasi-non –expansive mapping Defined in Hilbert space was proved in [4], convergence theorem of iterative schemes for non-expansive mapping have been presented and generalized in [5], Rhoades and Temir considered *t* and *i* self-maapping of *k*, where *t* is an-non-expansive mapping. they established the weak convergence of the sequence of Mann iterates to a common fixed point of *t* and *i*. in [15], Shahzas considered *t* and *i* non-self-mapping of k, where t is an i-non-expansive mapping they established the weak convergence of the sequence of modified Ishikawa iterates to a common fixed point of t.

2. PRELIMINARIES AND DEFINITION:

let e be a normed liner space *k*, a non-empty, convex subset of *e* and t a self map of k three most popular iteration procedures for obtaining fixed point of t if they exist are Mann iteration [6], defined by

$u1 \in K, u_n+1= (1-\alpha_n)\, u_n+\alpha_n T u_n$, n≥ 1. (1.1)

ISHIKAWA ITERATION [7], DEFINED BY,

$Z_{1\in k,} Z_{n+1} = (1-a_n)Z_n + a_n\, Y_n$ (1.2)

$y_n = (1 - \beta_n)z_n + \beta_n \mathrm{T} z_n$, n ≥ 1.

Noor iteration [40], defined b

V1 ∈ K, $v_{n+1} = (1-\alpha_n)\, v_n+\alpha_n\, \mathrm{T} w_n$ (1.3)

$w_n = (1-\beta_n)\, v_n+ \beta_n \mathrm{T} t_n$

$t_n = (1 - y_n)\, v_n + y_n \mathrm{T} t_n$, n ≥ 1

for certain choices of $\{\alpha_n\}$,$\{\beta_n\}$and $\{\gamma n\}$ ⊂ [0.1].

The multistep iteration [8], arbitrary fixed order p≥ 2definet by

$X_{n+1} = (1-\alpha_n)\, X_n + \alpha_n\, \mathrm{T} y_n^1$

$y_n^1 = (1- \beta_n^i)\, X_n + \beta_n^i\, \mathrm{t} y_n^{i+1}$, I = 1, 2, 3,…….

$y_n^{p-1} = (1-\beta_n^{p-1})\, X_n+ \beta_n^{p-1} \mathrm{T} x_n$ (1.4)

Where for all n∈N.

The sequence $\{\alpha_n\}$ is such that for all n∈N

$$\{\alpha_n\} \in (0,1),\ \lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = 0.$$

and for all n∈N.

$$\{\beta_n^i\} \text{ c } (0, 1),\ 1\geq\ i \geq\ p-1,$$

$$\lim_{n\to\infty} \beta_n^i = 0.$$

In the above taking p = 3in (1.4), we obtain iteration (1.3).

Taking p = 2 in (1.4), we obtain iteration (1.2).

Let k be a subset of normed linear space E = (E, ll. ll) and T self mapping of K then T is called non-expansive on k if

llTx – Tyll ≤ ll x – y ll. (1.5)

For all $x, y \in K, let\ F(T) = \{x \in K: Tx = x\}$

be denoted as the set of fixed point of a mapping T.

In [9], let k be a subset of normed linear space

E = (E, ll. ll) and T and I self mapping of K. Then T is called I-non expansive on K if

$ll\, Tx - Ty\, ll \ \leq\ lI\, Ix - Iy\, ll$ (1.6)

for all $x, y \in K$.

T is called I – quasi - non-expansive on K if

$ll\, Tx - f\, ll \leq\ ll\, Ix - f\, ll$ (1.7)

for all $x, y \in K$ and $f \in F(T) \cap F(I)$.

Let E be a real Banach space [10]. A subset K of E is said to be a retract of E if there exists a continuous map R:E → K such that Px = for all x ∈ K. A map P:E→E is said to be a relaction if P^2 =p. It follows that if a map P is a reaction, than Py = y For all y in the range of P. A set K is optimal if each point outside K can be move to be closer to all points of K. Note that every non-expansive retract is optimal in strictly convex Banach space optimal sets are closed and convex. However every closed convex subset of a Hilbert space is optimal and also a non-expansive retract.
Recall that a Banach space E is said to satisfy Opial's condition [11] if for each sequence $\{Xn\}$ in E, the condition $X_n \to X$ implies that

$$\lim_{n\to\infty} \|x_n - x\| < \lim_{n\to\infty} \|x_n - y\|$$

For all $y \in E$ with $y \neq x$.

The first non-linear Ergodic theorem was proved by Baillon [12] for general no-expansive mappings in Hilbert space H: if K is a closed and conves subset of h and T has a fixed point, then for every x ∈ K, $\{T^n x\}$ is weakly almost Convergent, as n x} is weakly almost Convergent, as n → ∞, to a fixed point of T. It was also

Shown by Pazy [16] that if H is a real Hilbert space and $\frac{1}{2}\sum_{i-0}^{n-1} t^i X$ converge weakly, as $n \to \infty\ to\ y \in K$, then $y \in F, (T)$.
The concept of a quasi- no- expansive mapping was initiated by Tricomi in 1914 for real function. Diaz and Metcalf [13] and Dotson [2] studies quasi – nonexpansive mapping ib Banach space. Recently this concept was given by Kirk [14] in metric space which we adapt to normed space as T is called a quasi - non-expansive aping provided

$$\|Tx - f\| \leq \|x - f\| \qquad (1.9)$$

For all x∈ K and ∈ F, (T).

Let E be a normed linear space, K be a non- empty convex subset of E with P as a non-expansive retraction. Let T: K → E be a given non-self mapping the modified multistep iteration scheme $\{X_n\}$ is defined by arbitrary fixed order p≥ 2.

$$X\_(n+1\ ) = P((l-\alpha\_n)\, X\_n + \alpha\_n\, Ty\_n\hat{}1)$$

$$y\_n\hat{}i = P((1-\beta\_n\hat{}i\, X\_n) + \beta\_n\hat{}iTy\_n\hat{}(i+1)), i = 1,2,\dots,p-2$$

$$y\_n\hat{}(p-1) = P((1-\beta\_n\hat{}(p-1))\, \alpha\_n + \beta\_n\hat{}(p-1)TX\_n$$

Where the sequence $\{\alpha_n\}$is such that for all $n = \varepsilon$

$$\{\alpha\_n\}c(0,1),\ 〚lim〛 \top(n \to \infty) = 0\ \sum\_(n=1)\hat{}\infty = \alpha$$

And for all $n \in N$

$$\{\beta_n^i\}\, c(0,1), 1 < i \leq p-1, \lim_{n\to\infty} \beta_n^i = 0$$

Clearly if T is self map then (α) reduces to an iteration scheme (1.4).

3 MAIN RESULTS:

Theorem: Let K be a closed convex bounded subset of uniformly convex Banach space E, which satisfies Opial's condition, and let T, l I non-self mapping of K with T an I – non expansive mapping, I a non-expansive mapping of K then for $Xo \in K$ the sequence $\{Xn\}$of modified multistep iterates converges weakly to common fixed point of $F(T) \cap F(I)$ is non-empty and a singleton then the proof is complete. We will assume that $F(T) \cap F(I)$ is not a singleton

$$II\, X\_(n+1) - f\, II = II\, p(1-\alpha\_n)\, Xn + (\alpha\_nTy\_n) - f\, II$$
$$\leq II\, (1-\alpha\_n)\, X\_n\, (\alpha\_nTy\_n\hat{}') - f\, II$$

$$\leq II\, (1-\alpha\_n)(X\_n - f) + \alpha\_n[\, p\, (1-\beta\_n\hat{},)\, X\_n + \beta\_n\hat{},Ty\_n\hat{}(2\,) - f\, II$$

$$\leq II\, (1-\alpha\_n)(X\_n - f) + \alpha\_n[(1-\beta\_n\hat{},)\, X\_n + \beta\_n\hat{},Ty\_n\hat{}(2\,) - f\, II$$

$$\leq (1-\alpha\_n)\, II\, (X\_n - f) + \alpha\_n\, II\, (1-\beta\_n\hat{},)\, X\_n + \beta\_n\hat{},Ty\_n\hat{}(2\,) - f\, II$$

$$\leq (1-\alpha\_n)\, II\ X\_n - f\, II + \alpha\_n\, II\, (1-\beta\_n\hat{},)\, (X\_n - f) + \beta\_n\hat{},[p(1$$
$$-\beta\_n\hat{}'2)\, X\_n] + \beta\_n\hat{}'2TX\_n - f\, II$$

$$\leq \ (1 - \alpha\_n)\, II \ X\_n - f\, II + \, II(1 - \beta\_n\^{,})\,(X\_n - f) \ + \beta\_n\^{,} [p(1 - \beta\_n\^{\prime}2)\, X\_n + \beta\_n\^{\prime}2TX\_n - f\, II$$

$$\leq \ (1 - \alpha\_n)\, II \ X\_n - f\, II\, \alpha\_n\, (1 - \beta\_n\^{,})\, II \ (X\_n - f) \ + \beta\_n\^{,}, \ II\, (1 - \beta\_n\^{\prime}2)\, X\_n + \ II\beta\_n\^2TX\_n - f\, II$$

$$\leq \ (1 - \alpha\_n)\, II \ X\_n - f\, II\, \alpha\_n\, (1 - \beta\_n\^{,})\, II \ (X\_n - f)\, II + \alpha\_n\, \beta\_n\^{,}, \ II(1 - \beta\_n\^{\prime}2)(X\_n - f) + \ \beta\_n\^{\prime}2(X\_n - f)\, II$$

$$\leq \ (1 - \alpha\_n)\, II \ X\_n - f\, II\, \alpha\_n\, (1 - \beta\_n\^{,})\, II \ (X\_n - f)\, II + \alpha\_n\, \beta\_n\^{,}, \ II\, (1 - \beta\_n\^2)$$

$$II\, (X\_n - f)\, II + \alpha\_n\, \beta\_n\^{,} (1 - \beta\_n\^2)\, II\, X\_n - \ f\, II \ = \ II\, X\_n - f\, II$$

For $\alpha\_n \neq 0$ and $\beta\_n\^i \ \neq \ 0\, \{II\ x\_n \ - \ f\ II\}$ is non-increasing sequence then $\lim_{n\to\infty} II\, X\_n - f\, II$ exists

Now we show that $\{X\_n\}$ converges weakly to a common fixed point of T and I the sequence $\{X_n\}$ contain a subsequence which converges weakly to a point in K Let $\{X_{nk}\}$ and $\{X_{mk}\}$be two subsequences of $\{X_n\}$ which converges weakly to f and q, respectively. We will show that f=g. Suppose that E satisfies Opial's condition ans that f $\neq$ q is in weak limit set of the sequence $\{X_n\}$. Then $\{X_{nk}\} \rightarrow$ and $\{X_{mk}\} \rightarrow$ q, respectively since $\lim_{n\to\infty}$ II X_n-f II exists for any f $\in$ F (T) $\cap$ F (I) by Opial's condition, we conclude that

$$lim\top(n \rightarrow \infty)\, II\, X\_n - f \ \ II \ = \ \lim_{n\to\infty} II \ \ X\_n \ - \ f \ \ II$$

$$< \ \lim_{n\to\infty} II \ \ X\_(nk\,) - \ q\, II$$

$$= \ \lim_{n\to\infty} II \ \ X\_mj - q\, II$$

$$< \ \lim_{n\to\infty} II \ \ 〚X〛\_mj - f \ \ II$$

$$= \ \lim_{n\to\infty} II \ \ X\_n \ - \ f \ \ II.$$

Tish is a contradiction. Thus, $\{X_n\}$ converges weakly to a common fixed point of F (T) $\cap$ F(I).

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