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Vertex Prime and Edge Ternion Sum Labeling of Certain Theta Related Graphs

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ABSTRACT:

A graph $G(V,E)$ is observed to admit Vertex Prime and Edge Ternion Sum Labeling when the vertices of the graph are labeled with unique integral values from $[1,|V|]$ in a way that for each edge uv, the end vertices u and v are designated labels that share no common positive factors except 1 and the edges of the graph are assigned labels with distinct integral values from $[1,|V|+|E|]$ such that the ternion sum of the designated labels of the vertices u,v and the edge uv constitutes uniform ternion sum or 1-edge varied ternion sum or k-edge varied ternion sum. In this research article we investigate that the uniform theta graph, quasi-uniform theta graph and subdivision of uniform theta graph admit Vertex Prime and Edge Ternion Sum Labeling.

Keywords: Uniform Theta Graph, Uniform Ternion Sum, $k-$Edge Varied Ternion Sum, Quasi Theta Graph, Ordered Theta Graph

1. Introduction and Preliminaries

In this research article we examine only finite, connected and non-oriented graphs with no multiple edges and loops. We denote the vertex set and the edge set by $V(G)$ and $E(G)$ of the graph G and the corresponding cardinality by $|V(G)|$ and $|E(G)|$ respectively. A *graph labeling* is an assignment of integers to the vertices or edges, or both subject to certain conditions [1]. One of the important areas in graph theory is graph labeling used in many applications like coding theory, x-ray crystallography, radar, astronomy, circuit design, communication network addressing, data base management. The notion of *prime labeling* was introduced by R. Entringer. Theta graph has its applications in bi-polar electric or magnetic fields. The field lines are represented by the longitudes of the theta graph and the distance between the North Pole and the South Pole influence the dipole moment. The labeling of the vertices and the edges with some conditions help in wireless communication through the radio waves based on the signal wave length. This mechanism may be considered in use by the researchers in different ways [2]. In this research article we investigate that the uniform theta graph, quasi-uniform theta graph and subdivision of uniform theta graph admit Vertex Prime and Edge Ternion Sum Labeling.

Definition 1.1. Generalized theta graph $\theta(L_1, L_2, L_3, \ldots, L_m)$ is obtained by joining two distinct isolated vertices with the end points of m disjoint paths $P_1, P_2, P_3, \ldots, P_m$ each of which has at least two internal vertices, where L_i, $1 \le i \le m$ denote the number of vertices on each of the path P_i. The internal paths are known as longitudes and the isolated vertex denoted by u joining one end of each longitude is known as the North Pole and the isolated vertex denoted by v joining the other end of each longitude is known as the South Pole [3].

Definition 1.2. A generalized theta graph is said to be a uniform theta graph if the number of vertices on each internal disjoint path are uniform. The uniform theta graph is denoted by$\theta(m, n)$, where m denotes the number of longitudes and n denotes the number of vertices on each of the longitudes [3].

Definition 1.3. A generalized theta graph is said to be a quasi-uniform theta graph if$L_1 = L_2 = L_3 = \cdots = L_{m-2} = L_{m-1} < L_m$, where$L_i$, $1 \le i \le m$ denote the number of vertices on each of the path P_i. Quasi-uniform theta graph is denoted by $\theta(m-1,1\colon n, n+t)$ where$t \ge 1$. $\theta(m-1,1\colon n, n+t)$ Represents quasi-uniform theta graph with n vertices on each $m-1$ longitudes $L_1, L_2, L_3, \ldots, L_{m-1}$ and $n+t$ vertices on the mth longitude L_m with $t \ge 1$ [3].

Definition 1.4. The Subdivision graph $S(G)$ of the graph G is obtained from G by adding a new vertex in the middle of every edge of G[4].

Definition 1.5. A graph $G = (V(G),\ E(G))$ is observed to admit prime labeling when the vertices of the graph are labeled with unique integral values from $[1,|V|]$ in a way that for every edge uv the labels designated to u and v share no common positive factors except 1[1].

2. Main definitions

Definition 2.1. Let $G(V, E)$ be a finite, connected and non-oriented graph. Let $V(G)$ and $E(G)$ be the vertex set and edge set with cardinality $|V(G)|$ and $|E(G)|$ respectively. We define the labeling of the vertices as a bijective mapping, $f\colon V(G) \to \{1,2,3,\ldots,|V(G)|\}$ such that for each edge $uv \in E(G)$, $gcf(f(u), f(v))$= 1 and we also define the labeling of the edges as an injective mapping, $g\colon E(G) \to \{1,2,3,\ldots,|V(G)| + |E(G)|\}$ such that each edge has a distinct label and for an edge $uv \in E(G)$, the **ternion sum** denoted by $\mathcal{T}(uv)$ is defined by $\mathcal{T}(uv) = f(u) + f(v) + g(uv)$.

Definition 2.2. The Uniform ternion sum of a graph G is the ternion sum $f(u) + f(v) + g(uv)$, which is a unique constant c for all the edges $uv \in E(G)$. The uniform ternion sum is denoted by$\mu(G) = c$.

Definition 2.3. The 1-Edge varied ternion sum of a graph G is the ternion sum $f(u) + f(v) + g(uv)$ which is a unique constant c for all $|E(G)| - 1$ edges and varies for only one edge of the graph$G$. The 1-Edge varied ternion sum is denoted by, $\mu_1(G) = \begin{cases} c & for\ |E(G)| - 1 \text{ edges} \\ c_1 & for\ 1 \text{ edge} \end{cases}$

Definition 2.4. The k −Edge varied ternion sum of a graph G is the ternion sum $f(u) + f(v) + g(uv)$ which is a unique constant c for $|E(G)| - k$ edges and another varied constant c_1 which is same for the remaining k edges of the graph G such that $|\mu_e(G)| \geq k$, where $\mu_e(G)$ denotes the set of edges that constitute unique ternion sum c and $|\mu_e(G)|$denotes the number of edges of the set $\mu_e(G)$. The k −Edge varied ternion sum is denoted by

$$\mu_k^1(G) = \begin{cases} c & for\ |E(G)| - k \text{ edges} \\ c_1 & for\ k \text{ edges} \end{cases} \text{ where } k > 1.$$

Definition 2.5. The k −Edge varied ternion sum with l −variations of a graph G is the ternion sum $f(u) + f(v) + g(uv)$ which is a unique constant c for $|E(G)| - k$ edges where $k = \sum_{i=1}^{l} k_i$ and another varied constant c_i which is same for the remaining k_i edges of the graph G such that $|\mu_e(G)| \geq k_i$ for $1 \leq i \leq l$ where $\mu_e(G)$ denotes the set of edges that constitute unique ternion sum c and $|\mu_e(G)|$denotes the number of edges of the set $\mu_e(G)$. The k −Edge varied ternion sum with l −variations are denoted by

$$\mu_k^l(G) = \begin{cases} c & for\ |E(G)| - k \text{ edges} \\ c_i & for\ k_i \text{ edges} \end{cases} \text{ where } k_i > 1 \text{ and } 1 \leq i \leq l, l \geq 1$$

Definition 2.6. A graph $G(V, E)$ is said to admit vertex prime and edge ternion sum labeling if it satisfies the following conditions:

1. The vertices of the graph are labeled with unique integral values from $[1, |V|]$ in a way that for each edge uv, the end vertices u and v are designated labels that share no common positive factors except 1.
2. The edges of the graph are assigned labels with distinct integral values from $[1, |V| + |E|]$ such that ternion sum is uniform ternion sum or 1 −Edge varied ternion sum or k −Edge varied ternion sum with l −variations

Note 1: In particular the k −Edge varied ternion sum is also known as k −Edge varied ternion sum with 1 −variation.

Note 2: If $|\mu_e(G)| < k_i$, $for\ 1 \leq i \leq l$ then the graph G is only a vertex prime graph and not an edge ternion sum graph.

Note 3: $\mu_e(G)$ denotes the set of edges that constitute unique ternion sum c and $|\mu_e(G)|$denotes the number of edges of the set $\mu_e(G)$.

Note 4: $\mu_e^1(G)$ denotes the set of edges that constitute ternion sum c_1 and $|\mu_e^1(G)|$ denotes the number of edges of the set $\mu_e^1(G)$.

3. Main results

3.1 Vertex prime and edge ternion sum labeling of uniform theta graph

Theorem 3.1.1:

The Uniform theta graph $\theta(m,n)$ admits vertex prime and edge ternion sum Labeling with $k-$edge varied ternion sum $\mu_k^1(\theta(m,n)) = \begin{cases} 2(mn+2) & for(\ |E(\theta(m,n))| - k)\ edges \\ mn+5 & for\ k\ edges \end{cases}$
for all $m,n \geq 3$ and $n \equiv 0(mod\ 2)$.

Proof:
Let $\theta(m,n)$ denote the uniform theta graph for all $m,n \geq 3$.
The vertex set $V(\theta(m,n)) = \{u, v, v_{i,j}\ /1 \leq i \leq m, 1 \leq j \leq n\ \}$ and $|V(\theta(m,n))| = mn + 2$
The edge set $E(\theta(m,n)) = \{uv_{i,1}\ /1 \leq i \leq m\} \cup \{vv_{i,n}\ /1 \leq i \leq m\} \cup \{v_{i,j}\ v_{i,j+1}/1 \leq i \leq m, 1 \leq j \leq n-1\}$ and $|E(\theta(m,n))| = m(n+1)$
We define the labeling of the vertices as a bijective mapping, $f: V(\theta(m,n)) \to \{1,2,3, \dots, mn+2\}$ by
$f(u) = 2$, $f(v) = 1$, $f(v_{i,j}) = n(i-1) + j + 2$, $1 \leq i \leq m$, $1 \leq j \leq n-1$
We define the labeling of the edges as an injective mapping, $g: E(\theta(m,n)) \to \{1,2,3, \dots, m(2n+1)+2\}$ by, for the edges $uv_{i,1}, 1 \leq i \leq m$, $g(\ uv_{i,1}) = \begin{cases} n(2m-i+1)-1 & if\ i \equiv 1(mod\ 2) \\ n(m-i+1) & if\ i \equiv 0(mod\ 2) \end{cases}$
For the edges $vv_{i,n}$, $1 \leq i \leq m$, $g(\ vv_{i,n}) = n(m-i)+2$
For the edges $v_{i,j}\ v_{i,j+1}$, $1 \leq i \leq m, 1 \leq j \leq n-1$, $g(v_{i,j}\ v_{i,j+1}) = 2(n(m-i+1)-j)-1$
Then for the edges $uv_{i,1}$, $vv_{i,n}$, and $v_{i,j}\ v_{i,j+1}$ we observe
$gcf\ (f(u), f(v_{i,1})) = 1$, $1 \leq i \leq m$, $gcf\left(f(v), f(vv_{i,n})\right) = 1$, $1 \leq i \leq m$
$gcf\left(f(v_{i,j}), f(\ v_{i,j+1})\right) = 1$, $1 \leq i \leq m$, $1 \leq j \leq n-1$
we also observe that for $1 \leq i \leq m$, the ternion sum for the edges $uv_{i,1}$ has two cases,

Case (1): $\boldsymbol{i \equiv 1(mod\ 2)}$
$\mathcal{T}(uv_{i,1}) = f(u) + f(v_{i,1}) + g(uv_{i,1}) = (2) + (\ n(i-1)+3) + (n(2m-i+1)-1)$
$= 2(mn+2)$, which is a unique constant,

Case (2): $\boldsymbol{i \equiv 0(mod\ 2)}$
$\mathcal{T}(uv_{i,1}) = f(u) + f(v_{i,1}) + g(uv_{i,1}) = (2) + (\ n(i-1)+3) + (n(m-i+1))$
$= mn+5$, which is a varied constant
we also observe that for $1 \leq i \leq m$, the ternion sum for the edges $vv_{i,n}$ is,
$\mathcal{T}(vv_{i,n}) = f(v) + f(v_{i,n}) + g(vv_{i,n}) = (1) + (n(i-1)+n+2) + (n(m-i)+2)$
$= mn+5$, which is a varied constant
For the edge $v_{i,j}\ v_{i,j+1}$, $1 \leq i \leq m, 1 \leq j \leq n-1$
$\mathcal{T}(v_{i,j}\ v_{i,j+1}) = f(v_{i,j}) + f(v_{i,j+1}) + g(v_{i,j}\ v_{i,j+1})$
$= (n(i-1)+j+2) + (n(i-1)+j+3) + (2(n(m-i+1)-j)-1)$
$= 2(mn+2)$. which is a unique constant,
we also infer that,

$\mu_e(\theta(m,n)) = \{uv_{i,1}/1 \le i \le m,\ \ i \equiv 1(mod\ 2)\} \cup \{\ v_{i,j}\ v_{i,j+1}/1 \le i \le m, 1 \le j \le n-1\}$

$$|\mu_e(\theta(m,n))| = \begin{cases} m(n-1) + \frac{m+1}{2} & if\ m \equiv 1(mod\ 2) \\ m(n-1) + \frac{m}{2} & if\ \ m \equiv 0(mod\ 2) \end{cases}$$

we also infer that

$\mu_e^1(\theta(m,n)) = \{uv_{i,1}/1 \le i \le m,\ \ i \equiv 0(mod\ 2)\} \cup \{uv_{i,n}/1 \le i \le m,\}$

$$k = |\mu_e^1(\theta(m,n))| = \begin{cases} \frac{3m-1}{2} & if\ m \equiv 1(mod\ 2) \\ \frac{3m}{2} & if\ \ m \equiv 0(mod\ 2) \end{cases}$$

We observe that in all the cases $|\ \mu_e(\theta(m,n))| > k$ for all $m,n \ge 3$ and $n \equiv 0(mod\ 2)$. Therefore, $\theta(m,n)$ admits vertex prime and edge ternion sum labeling with k −edge varied ternion sum

$$\mu_k^1(\theta(m,n)) = \begin{cases} 2(mn+2) & for(\ |E(\theta(m,n))| - k)\ edges \\ mn+5 & for\ k\ edges \end{cases}$$

for all $m,n \ge 3$ and $n \equiv 0(mod\ 2)$.

***Theorem 3.1.2*:**

The Uniform theta graph $\theta(m,n)$ admits vertex prime and edge ternion sum Labeling with uniform ternion sum$\mu(\theta(m,n)) = 2(mn+2)$, for all $m,n \ge 3$, $\ n \equiv 1(mod\ 2)$ and $mn+2$ is prime.

***Proof*:**

Let $\theta(m,n)$ denote the uniform theta graph for all$m,n \ge 3$, $n \equiv 1(mod\ 2)$ and $mn+2$ is prime.

The vertex set $V(\theta(m,n)) = \{u, v, v_{i,j}\ /1 \le i \le m, 1 \le j \le n\ \}$ and $|V(\theta(m,n))| = mn+2$

The edge set $E(\theta(m,n)) = \{uv_{i,1}\ /1 \le i \le m\} \cup \{vv_{i,n}\ /1 \le i \le m\} \cup \{v_{i,j}\ v_{i,j+1}/1 \le i \le m, 1 \le j \le n-1\}$ and $|E(\theta(m,n))| = m(n+1)$

We define the labeling of the vertices as a bijective mapping, $f: V(\theta(m,n)) \to \{1,2,3, \dots, mn+2\}$ by

$f(u) = 1$, $f(v) = mn+2$, $f(v_{i,j}) = n(i-1) + j + 1$, $1 \le i \le m$, $1 \le j \le n$

We define the labeling of the edges as an injective mapping, $g: E(\theta(m,n)) \to \{1,2,3, \dots, m(2n+1)+2\}$ by, for the edges$uv_{i,1}$, $1 \le i \le m$, $g(\ uv_{i,1}) = n(2m-i+1)+1$

For the edges$vv_{i,n}$, $1 \le i \le m$, $g(\ vv_{i,n}) = n(m-i)+1$

For the edges$v_{i,j}\ v_{i,j+1}$, $1 \le i \le m, 1 \le j \le n-1$, $g(v_{i,j}\ v_{i,j+1}) = 2(n(m-i+1)-j)-1$

Then for the edges$uv_{i,1}$, $vv_{i,n}$, and $v_{i,j}\ v_{i,j+1}$ we observe

$gcf\ (f(u), f(v_{i,1})) = 1$, $1 \le i \le m$, $\ gcf\left(f(v), f\left(vv_{i,n}\right)\right) = 1$, $1 \le i \le m$

$gcf\left(f\left(v_{i,j}\right), f\left(\ v_{i,j+1}\right)\right) = 1$, $1 \le i \le m$, $1 \le j \le n-1$

we also observe that for $1 \le i \le m$, the ternion sum for the edges $uv_{i,1}$ is,

$\mathcal{T}\left(uv_{i,1}\right) = f(u) + f\left(v_{i,1}\right) + g\left(uv_{i,1}\right) = (1) + (\ n(i-1)+2) + (n(2m-i+1)+1)$

$= 2(mn+2)$, which is a unique constant

we also observe that for $1 \le i \le m$, the ternion sum for the edges $vv_{i,n}$ is,

$\mathcal{T}\left(vv_{i,n}\right) = f(\ v) + f\left(v_{i,n}\right) + g\left(vv_{i,n}\right) = (mn+2) + (n(i-1)+n+1) + (n(m-i)+1)$

$= 2(mn+2)$, which is a unique constant

For the edge $v_{i,j}\ v_{i,j+1}$, $1 \le i \le m, 1 \le j \le n-1$

$\mathcal{T}\left(v_{i,j}\ v_{i,j+1}\right) = f\left(v_{i,j}\right) + f\left(v_{i,j+1}\right) + g\left(v_{i,j}\ v_{i,j+1}\right)$

$= (n(i-1)+j+1) + (n(i-1)+j+2) + (2(n(m-i+1)-j)-1)$
$= 2(mn+2)$, which is a unique constant

we also infer that, $\mu_e(\theta(m,n)) = \{uv_{i,1} / 1 \le i \le m\} \cup \{vv_{i,n} / 1 \le i \le m\} \cup \{v_{i,j}\, v_{i,j+1} / 1 \le i \le m, 1 \le j \le n-1\}$ and $|\mu_e(\theta(m,n))| = m(n+1)$

We observe that $|\mu_e(\theta(m,n))| = |E(\theta(m,n))|$ for all $m, n \ge 3,\ n \equiv 1(mod\ 2)$ and $mn+2$ is prime. Therefore, $\theta(m,n)$ admits vertex prime and edge ternion sum labeling with uniform ternion sum, $\mu(\theta(m,n)) = 2(mn+2)$

The uniform theta graph and its vertex prime and edge ternion sum labeling are as in the following figures 3.1.1., 3.1.2. and 3.1.3. respectively.

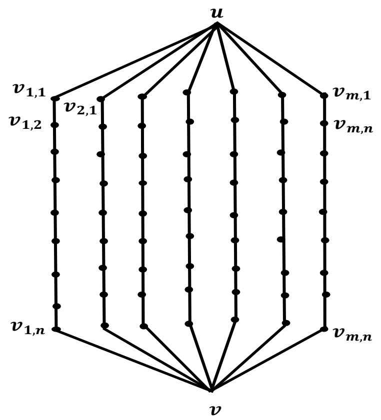

Figure 3.1.1. Uniform theta graph $\theta(m,n)$

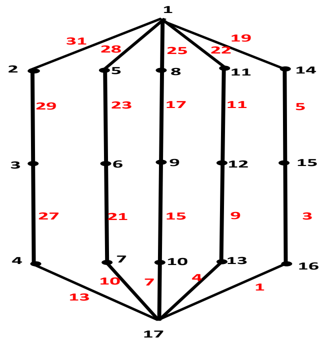

Figure 3.1.2. Uniform theta graph $\theta(5,3)$ with uniform ternion sum $\mu(\theta(m,n)) = 34$

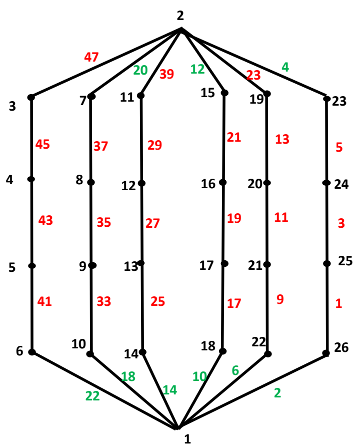

Figure 3.1.3. Uniform theta graph $\theta(6,4)$ with varied ternion sum

$$\mu_k^1(\theta(m,n)) = \begin{cases} 52 & for\ 21\ edges \\ 29 & for\ 9\ edges \end{cases}$$

3.4 **Vertex prime and edge ternion sum labeling of quasi-uniform theta graph**

Theorem 3.2.1:

The quasi-uniform theta graph $\theta(m-1,n:1,n+t)$ admits vertex prime and edge ternion sum Labeling with k −edge varied ternion sum

$$\mu_k^1(\theta(m,n)) = \begin{cases} 2(mn+t+2) & for(\ |E(\theta(m,n))|-k)\ edges \\ \left\{\begin{cases} mn+t+5 & if\ \ t \equiv 0(mod\ 2) \\ n(m-1)+7\ if\ \ m \equiv 0(mod\ 2) and\ \ t \equiv 1(mod\ 2) \\ n(m-1)+5\ if\ \ m \equiv 1(mod\ 2) and\ \ t \equiv 1(mod\ 2) \end{cases} for\ k\ edges\right\} \end{cases}$$

for all $m,n \geq 3$ and $n \equiv 0(mod\ 2)$.

Proof:

Let $\theta(m-1,n:1,n+t)$ denote the quasi-uniform theta graph for all $m,n \geq 3$ and $t \geq 1$.

The vertex set $V(\theta(m-1,n:1,n+t)) =$
$\{u,v,v_{i,j}\ /1 \leq i \leq m-1, 1 \leq j \leq n\} \cup \{\ v_{m,j}\ /\ 1 \leq j \leq n+t, t \geq 1\ \}$ and $|V(\theta(m-1,n:1,n+t))| = mn+t+2$

The edge set $E(\theta(m-1,n:1,n+t)) = \{uv_{i,1}\ /1 \leq i \leq m\} \cup \{vv_{i,n}\ /1 \leq i \leq m-1\} \cup \{vv_{m,n+t}\} \cup \{v_{i,j}\ v_{i,j+1}/1 \leq i \leq m, 1 \leq j \leq n+t-1\}$ and
$|E(\theta(m-1,n:1,n+t))| = m(n+1)+t$

We define the labeling of the vertices as a bijective mapping, $f: V(\theta(m-1,n:1,n+t)) \rightarrow \{1,2,3,\ldots,mn+t+2\}$ by $f(u) = 2$, $f(v) = 1$, $f(v_{i,j}) = n(i-1)+j+2$, $1 \leq i \leq m$, $1 \leq j \leq n+t$

We define the labeling of the edges as an injective mapping, $g: E(\theta(m-1,n:1,n+t)) \rightarrow \{1,2,3,\ldots,m(2n+1)+2t+2\}$ by

For the edges $uv_{i,1}$, $1 \leq i \leq m$ we have three cases

Case (1): $m \equiv 0,1(mod\ 2)$ and $t \equiv 0(mod\ 2)$ for all $m \geq 3$ and $t \geq 1$ except $t = 2$ and when $m \equiv 0(mod\ 2)$, $g(\ uv_{i,1}) = \begin{cases} n(2m-i+1)+2t-1 & if\ i \equiv 1(mod\ 2) \\ n(m-i+1)+t & if\ i \equiv 0(mod\ 2) \end{cases}$

Case (2): $m \equiv 0(mod\ 2)$ and $t \equiv 1(mod\ 2)$ for all $m \geq 3$ and $t \geq 1$ and also $t = 2$ when $m \equiv 0(mod\ 2)$, $g(\ uv_{i,1}) = \begin{cases} n(2m-i+1)+2t-1 & if\ i \equiv 1(mod\ 2) \\ n(m-i)+2 & if\ i \equiv 0(mod\ 2) \end{cases}$

Case (3): $m \equiv 1(mod\ 2)$ and $t \equiv 1(mod\ 2)$ for all $m \geq 3$ and $t \geq 1$

$g(\ uv_{i,1}) = \begin{cases} n(2m-i+1)+2t-1 & if\ i \equiv 1(mod\ 2) \\ n(m-i) & if\ i \equiv 0(mod\ 2) \end{cases}$

For the edges $vv_{i,n}$, $1 \leq i \leq m-1$ we have three cases

Case (1): $m \equiv 0,1(mod\ 2)$ and $t \equiv 0(mod\ 2)$ for all $m \geq 3$ and $t \geq 1$ except $t = 2$ and when $m \equiv 0(mod\ 2)$, $g(\ vv_{i,n}) = n(m-i)+t+2$

Case (2): $m \equiv 0(mod\ 2)$ and $t \equiv 1(mod\ 2)$ for all $m \geq 3$ and $t \geq 1$ and also $t = 2$ when $m \equiv 0(mod\ 2)$, $g(\ vv_{i,n}) = n(m-i-1)+4$

Case (3): $m \equiv 1(mod\ 2)$ and $t \equiv 1(mod\ 2)$ for all $m \geq 3$ and $t \geq 1$, $g\left(vv_{i,n}\right) = n(m-i-1)+2$
For the edge $vv_{m,n+t}$, we have two cases

Case (1): $m \equiv 0,1(mod\ 2)$ and $t \equiv 0(mod\ 2)$ for all $m \geq 3$ and $t \geq 1$ except $t = 2$ when $m \equiv 0(mod\ 2)$, $g\left(vv_{m,n+t}\right) = 2$

Case (2): $m \equiv 0,1(mod\ 2)$ and $t \equiv 1(mod\ 2)$ for all $m \geq 3$ and $t \geq 1$ and also $t = 2$ when $m \equiv 0(mod\ 2)$, $g(vv_{m,n+t}) = mn+t+1$
For the edges $v_{i,j}\, v_{i,j+1}, 1 \leq i \leq m, 1 \leq j \leq n+t-1,\ g(v_{i,j}\, v_{i,j+1}) = 2(n(m-i+1)+t-j)-1$

Then for the edges $uv_{i,1}$, $vv_{i,n}$, $vv_{m,n+t}$, $v_{i,j}\, v_{i,j+1}$ we observe,
$gcf\ (f(u), f(v_{i,1})) = 1$ for $1 \leq i \leq m$, $gcf\left(f(v), f\left(vv_{i,n}\right)\right) = 1$ for $1 \leq i \leq m-1$
$gcf\left(f(v), f\left(v_{m,n+t}\right)\right) = 1$, $gcf\left(f\left(v_{i,j}\right), f\left(v_{i,j+1}\right)\right) = 1$ for $1 \leq i \leq m$, $1 \leq j \leq n+t-1$
we also observe that for $1 \leq i \leq m$, the ternion sum for the edges $uv_{i,1}$ has two cases,

Case (1): $m \equiv 0,1(mod\ 2)$ and $t \equiv 0(mod\ 2)$ for all $m \geq 3$ and $t \geq 1$ except $t = 2$ when $m \equiv 0(mod\ 2)$

Sub Case (1): $\boldsymbol{i \equiv 1(mod\ 2)}$
$\mathcal{T}\left(uv_{i,1}\right) = f(u) + f\left(v_{i,1}\right) + g\left(uv_{i,1}\right) = 2(mn+t+2)$, which is a unique constant,

Sub Case (2): $\boldsymbol{i \equiv 0(mod\ 2)}$
$\mathcal{T}\left(uv_{i,1}\right) = f(u) + f\left(v_{i,1}\right) + g\left(uv_{i,1}\right) = mn+t+5$, which is a varied constant

Case (2): $m \equiv 0(mod\ 2)$ and $t \equiv 1(mod\ 2)$ for all $m \geq 3$ and $t \geq 1$ and also $t = 2$ when $m \equiv 0(mod\ 2)$

Sub Case (1): $\boldsymbol{i \equiv 1(mod\ 2)}$
$\mathcal{T}\left(uv_{i,1}\right) = f(u) + f\left(v_{i,1}\right) + g\left(uv_{i,1}\right) = 2(mn+t+2)$, which is a unique constant,

Sub Case (2): $\boldsymbol{i \equiv 0(mod\ 2)}$
$\mathcal{T}\left(uv_{i,1}\right) = f(u) + f\left(v_{i,1}\right) + g\left(uv_{i,1}\right) = n(m-1)+7$, which is a varied constant

Case (3): $m \equiv 1(mod\ 2)$ and $t \equiv 1(mod\ 2)$ for all $m \geq 3$ and $t \geq 1$

Sub Case (1): $\boldsymbol{i \equiv 1(mod\ 2)}$
$\mathcal{T}\left(uv_{i,1}\right) = f(u) + f\left(v_{i,1}\right) + g\left(uv_{i,1}\right) = 2(mn+t+2)$, which is a unique constant,

Sub Case (2): $\boldsymbol{i \equiv 0(mod\ 2)}$
$\mathcal{T}\left(uv_{i,1}\right) = f(u) + f\left(v_{i,1}\right) + g\left(uv_{i,1}\right) = n(m-1)+5$, which is a varied constant
we also observe that for $1 \leq i \leq m$, the ternion sum for the edges $vv_{i,n}$ is,

Case (1): $m \equiv 0,1(mod\ 2)$ and $t \equiv 0(mod\ 2)$ for all $m \geq 3$ and $t \geq 1$ except $t = 2$ when $m \equiv 0(mod\ 2)$, $\mathcal{T}(vv_{i,n}) = f(v) + f(v_{i,n}) + g(vv_{i,n}) = mn + t + 5$, which is a varied constant

Case (2): $m \equiv 0(mod\ 2)$ and $t \equiv 1(mod\ 2)$ for all $m \geq 3$ and $t \geq 1$ and also $t = 2$ when $m \equiv 0(mod\ 2)$, $\mathcal{T}(vv_{i,n}) = f(v) + f(v_{i,n}) + g(vv_{i,n}) = n(m-1) + 7$, which is a varied constant

Case (3): $m \equiv 1(mod\ 2)$ and $t \equiv 1(mod\ 2)$ for all $m \geq 3$ and $t \geq 1$
$\mathcal{T}(vv_{i,n}) = f(v) + f(v_{i,n}) + g(vv_{i,n}) = n(m-1) + 5$, which is a varied constant
we also observe that the ternion sum for the edges $vv_{m,n+t}$ is,

Case (1): $m \equiv 0,1(mod\ 2)$ and $t \equiv 0(mod\ 2)$ for all $m \geq 3$ and $t \geq 1$ except $t = 2$ when $m \equiv 0(mod\ 2)$, $\mathcal{T}(vv_{m,n+t}) = f(v) + f(v_{m,n+t}) + g(vv_{m,n+t}) = mn + t + 5$, which is a varied constant

Case (2): $m \equiv 0,1(mod\ 2)$ and $t \equiv 1(mod\ 2)$ for all $m \geq 3$ and $t \geq 1$ and also $t = 2$ when $m \equiv 0(mod\ 2)$
$\mathcal{T}(vv_{m,n+t}) = f(v) + f(v_{m,n+t}) + g(vv_{m,n+t}) = 2(mn + t + 2)$, which is a unique constant
For the edge $v_{i,j}\ v_{i,j+1}$, , $1 \leq i \leq m, 1 \leq j \leq n + t - 1$
$\mathcal{T}(v_{i,j}\ v_{i,j+1}) = f(v_{i,j}) + f(v_{i,j+1}) + g(v_{i,j}\ v_{i,j+1}) = 2(mn + t + 2)$, which is a unique constant

We observe that in all the cases
$|\mu_e(\theta(m-1, n{:}\, 1, n+t))| > |\mu'_e(\theta(m-1, n{:}\, 1, n+t))|$ for all $m, n \geq 3$ and $n \equiv 0(mod\ 2)$. Therefore, $\theta(m-1, n{:}\, 1, n+t)$ admits vertex prime and edge ternion sum labeling for all $m, n \geq 3$ and $n \equiv 0(mod\ 2)$.

Theorem 3.2.2:
The quasi-uniform theta graph $\theta(m-1, n{:}\, 1, n+t)$ admits vertex prime and edge ternion sum Labeling with with k −edge varied ternion sum

$$\mu_k^1(\theta(m-1, n{:}\, 1, n+t)) = \begin{cases} 2(mn + t + 2) & for(\ |E(\theta(m-1, n{:}\, 1, n+t))| - k)\ edges \\ 2n(m-1) + t + 2 & for\ k\ edges \end{cases}$$

for all $m, n \geq 3$, $t \geq 2, n \equiv 1(mod\ 2)$, $m \equiv 1(mod\ 2), t \equiv 0(mod\ 2)$ and $mn + t + 2$ is prime.

Proof:
Let $\theta(m-1, n{:}\, 1, n+t)$ denote the quasi-uniform theta graph for all $m, n \geq 3$, $t \geq 2, n \equiv 1(mod\ 2)$, $m \equiv 1(mod\ 2), t \equiv 0(mod\ 2)$. The vertex set $V(\theta(m-1, n{:}\, 1, n+t)) = \{u, v, v_{i,j}\ /1 \leq i \leq m-1, 1 \leq j \leq n\} \cup \{\ v_{m,j}\ /\ 1 \leq j \leq n+t, t \geq 1\}$ and $|V(\theta(m-1, n{:}\, 1, n+t))| = mn + t + 2$ and the edge set $E(\theta(m-1, n{:}\, 1, n+t)) = \{uv_{i,1}\ /1 \leq i \leq m\} \cup \{vv_{i,n}\ /1 \leq i \leq m-1\} \cup \{vv_{m,n+t}\} \cup \{v_{i,j}\ v_{i,j+1}/1 \leq i \leq m, 1 \leq j \leq n+t-1\}$ and
$|E(\theta(m-1, n{:}\, 1, n+t))| = m(n+1) + t$
We define the labeling of the vertices as a bijective mapping, $f{:}\, V(\theta(m-1, n{:}\, 1, n+t)) \rightarrow \{1,2,3, \ldots, mn + t + 2\}$ by, $f(u) = 1$, $f(v) = mn + t + 2$
$f(v_{i,j}) = n(i-1) + j + 1$, $1 \leq i \leq m$, $1 \leq j \leq n + t$

We define the labeling of the edges as an injective mapping, $g: E(\theta(m-1, n: 1, n+t)) \to \{1,2,3, \dots, m(2n+1)+2t+2\}$ by, for the edges $uv_{i,1}, 1 \le i \le m$
$g(uv_{i,1}) = n(2m-i+1)+2t+1$
For the edges $vv_{i,n}$, $1 \le i \le m-1$, $g(vv_{i,n}) = \begin{cases} n(m-i)+t+1 & if\ i \equiv 0(mod\ 2) \\ n(m-i-2)+2 & if\ i \equiv 1(mod\ 2) \end{cases}$
For the edge $vv_{m,n+t}$, $g(vv_{m,n+t}) = 1$
For the edges $v_{i,j}\ v_{i,j+1}, 1 \le i \le m, 1 \le j \le n+t-1$, $g(v_{i,j}\ v_{i,j+1}) = 2(n(m-i+1)+t-j)+1$
Then for the edges $uv_{i,1}$, $vv_{i,n}$, $vv_{m,n+t}$, $v_{i,j}\ v_{i,j+1}$ we observe,
$gcf\ (f(u), f(v_{i,1})) = 1$ for $1 \le i \le m$, $gcf\left(f(v), f(vv_{i,n})\right) = 1$ for $1 \le i \le m-1$
$gcf\left(f(v), f(v_{m,n+t})\right) = 1$, $gcf\left(f(v_{i,j}), f(v_{i,j+1})\right) = 1$ for $1 \le i \le m$, $1 \le j \le n+t-1$
we also observe that for $1 \le i \le m$, the ternion sum for the edges $uv_{i,1}$ is,
$\mathcal{T}(uv_{i,1}) = f(u) + f(v_{i,1}) + g(uv_{i,1}) = 2(mn+t+2)$, which is a unique constant
we also observe that for $1 \le i \le m$, the ternion sum for the edges $vv_{i,n}$ has two cases,

Case (1): $if\ i \equiv 0(mod\ 2),\ 1 \le i \le m-1$
$\mathcal{T}(vv_{i,n}) = f(v) + f(v_{i,n}) + g(vv_{i,n}) = 2(mn+t+2)$, which is a unique constant

Case (2): $if\ i \equiv 1(mod\ 2),\ 1 \le i \le m-1$
$\mathcal{T}(vv_{i,n}) = f(v) + f(v_{i,n}) + g(vv_{i,n}) = 2n(m-1)+t+2$, which is a varied constant
we also observe that the ternion sum for the edges $vv_{m,n+t}$ is,
$\mathcal{T}(vv_{m,n+t}) = f(v) + f(v_{m,n+t}) + g(vv_{m,n+t}) = 2(mn+t+2)$, which is a unique constant
For the edge $v_{i,j}\ v_{i,j+1}$, , $1 \le i \le m, 1 \le j \le n+t-1$
$\mathcal{T}(v_{i,j}\ v_{i,j+1}) = f(v_{i,j}) + f(v_{i,j+1}) + g(v_{i,j}\ v_{i,j+1}) = 2(mn+t+2)$, which is a unique constant
we also infer that, $\mu_e(\theta(m-1, n: 1, n+t)) = \{uv_{i,1}/1 \le i \le m\} \cup \{v_{i,j}\ v_{i,j+1}/1 \le i \le m, 1 \le j \le n+t-1\} \cup \{vv_{i,n}/1 \le i \le m-1,\ i \equiv 0(mod\ 2)\} \cup \{vv_{m,n+t}\}$
$|\mu_e(\theta(m-1, n: 1, n+t))| = \frac{m(2n+1)+2t+1}{2}$
we also infer that $\mu_e^1(\theta(m-1, n: 1, n+t)) = \{vv_{i,n}/1 \le i \le m-2,\ i \equiv 1(mod\ 2)\}$
$k = |\mu_e^1(\theta(m-1, n: 1, n+t))| = \frac{m-1}{2}$
We observe that $|\mu_e(\theta(m-1, n: 1, n+t))| > k$ for all $m, n \ge 3$ and $n \equiv 1(mod\ 2)$. Therefore, $\theta(m-1, n: 1, n+t)$ admits vertex prime and edge ternion sum labeling with k −edge varied ternion sum $\mu_k^1(\theta(m-1, n: 1, n+t)) = \begin{cases} 2(mn+t+2) & for(|E(\theta(m-1, n: 1, n+t))| - k)\ edges \\ 2n(m-1)+t+2 & for\ k\ edges \end{cases}$ for all $m, n \ge 3$, $t \ge 2, n \equiv 1(mod\ 2)$, $m \equiv 1(mod\ 2), t \equiv 0(mod\ 2)$ and $mn+t+2$ is prime.

Theorem 3.2.3:
The quasi-uniform theta graph $\theta(m-1, n: 1, n+t)$ admits vertex prime and edge ternion sum labeling with 1 −edge varied ternion sum

$$\mu_1^1(\theta(m-1, n: 1, n+t)) = \begin{cases} 2(mn+t+2) & for(|E(\theta(m-1, n: 1, n+t))| - 1)\ edges \\ n(2m-1)+t+5 & for\ 1\ edges \end{cases}$$

for all $m, n \ge 3$, $t \ge 2, n \equiv 1(mod\ 2)$, $m \equiv 0(mod\ 2), t \equiv 1(mod\ 2)$ and $mn+t+2$ is prime.

Proof:

Let $\theta(m-1,n:1,n+t)$ denote the quasi-uniform theta graph for all $m,n \geq 3,\ t \geq 2, n \equiv 1(mod\ 2),\ m \equiv 1(mod\ 2), t \equiv 0(mod\ 2)$.

The vertex set $V\big(\theta(m-1,n:1,n+t)\big) = \{u,v,v_{i,j}\ /1 \leq i \leq m-1, 1 \leq j \leq n\} \cup \{\ v_{m,j}\ / 1 \leq j \leq n+t, t \geq 1\ \}$ and $|V(\theta(m-1,n:1,n+t))| = mn+t+2$. The edge set $E(\theta(m-1,n:1,n+t)) = \{uv_{i,1}\ /1 \leq i \leq m\} \cup \{vv_{i,n}\ /1 \leq i \leq m-1\} \cup \{vv_{m,n+t}\} \cup \{v_{i,j}\ v_{i,j+1}/ 1 \leq i \leq m, 1 \leq j \leq n+t-1\}$ and $|E(\theta(m-1,n:1,n+t))| = m(n+1)+t$

We define the labeling of the vertices as a bijective mapping, $f: V(\theta(m-1,n:1,n+t)) \rightarrow \{1,2,3,\dots,mn+t+2\}$ by, $f(u) = 1$, $f(v) = mn+t+2$

$f(v_{i,j}) = n(i-1)+j+1,\ 1 \leq i \leq m,\ 1 \leq j \leq n+t$

We define the labeling of the edges as an injective mapping, $g: E(\theta(m-1,n:1,n+t)) \rightarrow \{1,2,3,\dots,m(2n+1)+2t+2\}$ by, for the edges $uv_{i,1}$, $1 \leq i \leq m$

$g(\ uv_{i,1}) = n(2m-i+1)+2t+1$

For the edges $vv_{i,n}$, $1 \leq i \leq m-2$, $g(\ vv_{i,n}) = n(m-i)+t+1$

For the edge $vv_{m-1,n}$, $g(\ vv_{m-1,n}) = 2$, For the edge $vv_{m,n+t}$, $g(\ vv_{m,n+t}) = 1$

For the edges $v_{i,j}\ v_{i,j+1}$, $1 \leq i \leq m, 1 \leq j \leq n+t-1$, $g(v_{i,j}\ v_{i,j+1}) = 2(n(m-i+1)+t-j)+1$

Then for the edges $uv_{i,1}$, $vv_{i,n}$, $vv_{m,n+t}$, $v_{i,j}\ v_{i,j+1}$ we observe,

$gcf\ (f(u), f(v_{i,1})) = 1$ for $1 \leq i \leq m$, $gcf\left(f(v), f\left(vv_{i,n}\right)\right) = 1$ for $1 \leq i \leq m-1$

$gcf\left(f(v), f\left(v_{m,n+t}\right)\right) = 1$, $gcf\left(f\left(v_{i,j}\right), f\left(\ v_{i,j+1}\right)\right) = 1$, $1 \leq i \leq m$, $1 \leq j \leq n+t-1$

we also observe that, the ternion sum for the edges $\{uv_{i,1}, 1 \leq i \leq m\}$, $\{vv_{i,n}\ , 1 \leq i \leq m-2\}$, $\{vv_{m,n+t}\}$, $\{v_{i,j}\ v_{i,j+1}, 1 \leq i \leq m, 1 \leq j \leq n+t-1\}$ is $\mathcal{T}\left(uv_{i,1}\right) = 2(mn+t+2)$, which is a unique constant

we also observe that the ternion sum for the edge $vv_{m-1,n}$ is $\mathcal{T}\left(vv_{m-1,n}\right) = f(v) + f\left(v_{m-1,n}\right) + g\left(vv_{m-1,n}\right) = n(2m-1)+t+5$, which is a varied constant

$\mu_e(\theta(m,n)) = \{uv_{i,1}/1 \leq i \leq m\} \cup \{\ v_{i,j}\ v_{i,j+1}/1 \leq i \leq m, 1 \leq j \leq n+t-1\} \cup \{vv_{i,n}/ 1 \leq i \leq m-2\} \cup \{vv_{m,n+t}\}$

$|\mu_e(\theta(m-1,n:1,n+t))| = m(n+1)+t-1 = |E(\theta(m-1,n:1,n+t))| - 1$

we also infer that, $\mu_e^1(\theta(m-1,n:1,n+t)) = \{vv_{m-1,n}\}$ and $k = |\mu_e^1(\theta(m-1,n:1,n+t))| = 1$

We observe that $|\ \mu_e(\theta(m-1,n:1,n+t))| > k$ for all $m,n \geq 3$ and $n \equiv 1(mod\ 2)$.

Therefore, the quasi-uniform theta graph $\theta(m-1,n:1,n+t)$ admits vertex prime and edge ternion sum labeling with $1-$edge varied ternion sum

$$\mu_1^1(\theta(m-1,n:1,n+t)) = \begin{cases} 2(mn+t+2) & for(\ |E(\theta(m-1,n:1,n+t))| - 1)\ edges \\ n(2m-1)+t+5 & for\ 1\ edges \end{cases}$$

for all $m,n \geq 3,\ t \geq 2, n \equiv 1(mod\ 2),\ m \equiv 0(mod\ 2), t \equiv 1(mod\ 2)$ and $mn+t+2$ is prime.

The quasi- uniform theta graph and its vertex prime and edge ternion sum labeling are as in the following figures 3.2.1. and 3.2.2. respectively.

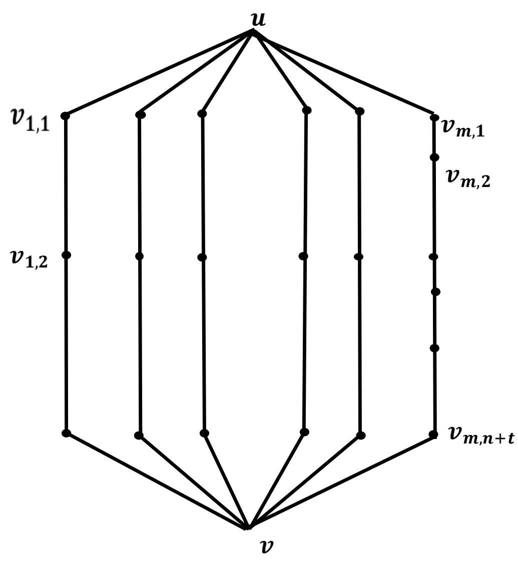

Figure 3.2.1. Quasi-uniform theta graph $\theta(m-1, n: 1, n+t)$

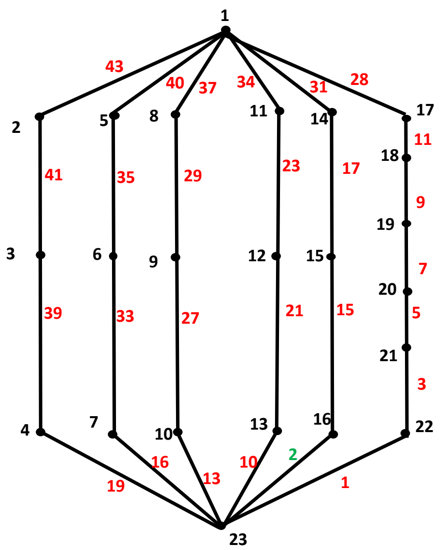

Figure 3.2.2. Quasi-uniform theta graph $\theta(5,3: 1,6)$ with 1-edge varied ternion sum

$$\mu_1^1(\theta(5,3:1,6)) = \begin{cases} 46 & for\ 26\ edges \\ 41 & for\ 1\ edge \end{cases}$$

3.5 Vertex prime and edge ternion sum labeling of subdivision of uniform theta graph

Theorem 3.3.1:

The subdivision of uniform theta graph $S[\theta(m,n)]$ admits vertex prime and edge ternion sum labeling with uniform ternion sum $\mu(S[\theta(m,n)]) = 2(m(2n+1)+2)$, for all $m, n \geq 3$ and $m(2n+1)+2$ is prime.

Proof:

Let $\theta(m,n)$ denote the uniform theta graph for all $m, n \geq 3$,

The vertex set $V(\theta(m,n)) = \{u, v, v_{i,j} / 1 \leq i \leq m, 1 \leq j \leq n\}$ and $|V(\theta(m,n))| = mn + 2$

The edge set $E(\theta(m,n)) = \{uv_{i,1} /1 \leq i \leq m\} \cup \{vv_{i,n} /1 \leq i \leq m\} \cup \{v_{i,j}\, v_{i,j+1}/1 \leq i \leq m, 1 \leq j \leq n-1\}$ and $|E(\theta(m,n))| = m(n+1)$
Let $S[\theta(m,n)]$ denote the subdivision of uniform theta graph for all $m, n \geq 3$,
The vertex set $V(S[\theta(m,n)]) = \{u, v, v_{i,j} /1 \leq i \leq m, 1 \leq j \leq 2n+1\}$ and $|V(S[\theta(m,n)])| = m(2n+1)+2$. The edge set $E(S[\theta(m,n)]) = \{uv_{i,1} /1 \leq i \leq m\} \cup \{vv_{i,n} /1 \leq i \leq m\} \cup \{v_{i,j}\, v_{i,j+1}/1 \leq i \leq m, 1 \leq j \leq 2n\}$ and $|E(\theta(m,n))| = 2m(n+1)$
From theorem 3.1.2. we observe that the uniform theta graph $\theta(p,q)$ admits vertex prime and edge ternion sum Labeling with uniform ternion sum $\mu(\theta(p,q)) = 2(pq+2)$, for all $p, q \geq 3$, $q \equiv 1(mod\ 2)$ and $pq+2$ is prime. Here we observe that the subdivision graph of theta graph is a theta graph with $q = 2n+1$ and $p = m$ and since $(2n+1) \equiv 1(mod\ 2)$ we have $\mu(S[\theta(m,n)]) = 2(m(2n+1)+2)$ and hence the subdivision of uniform theta graph $S[\theta(m,n)]$ admits vertex prime and edge ternion sum labeling with uniform ternion sum $\mu(S[\theta(m,n)]) = 2(m(2n+1)+2)$, for all $m, n \geq 3$, and $m(2n+1)+2$ is prime.

Corollary 3.3.1: If $\theta(m,n)$ is the uniform theta graph then
$\mu(S[\theta(m,n)]) = \mu(\theta(m,n)) + 2|E(\theta(m,n))|$.

3. Conclusion

In this research article we investigate that the uniform theta graph, quasi-uniform theta graph and subdivision of uniform theta graph admit vertex prime and edge ternion sum labeling. Motivated by the concept of generalized theta graph, in our future work we will be constructing a new type of theta graph known as ordered theta graph, even-ordered theta graph and odd-ordered theta graph and also investigate vertex prime and edge ternion sum labeling on the newly constructed graphs.

4. References

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