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Domination in Edge Product Cluster Hypergraphs

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abstract

The study on Domination in Edge Product Cluster Hypergraph aims for a balance between the concepts of Domination and Edge Product in Cluster Hypergraphs. The cluster hypergraph $H = (V_X, E)$ is said to be an Edge Product Cluster Hypergraph if there exists an edge function $f: E \to I$ such that the edge function f and the corresponding edge product function F of f on V_X have the following conditions are $F(v) \in I$ for every $v \in V_X$ and if $f(e_1) \times f(e_2) \times \ldots \times f(e_i) \in I$ for some edges $e_1, e_2, \ldots e_i \in E(H)$ then the edges $e_1, e_2, \ldots e_i$ are all adjacent to some vertex V_X . This article explores the concept of Domination in Edge Product Cluster Hypergraphs. It is demonstrated that every dominating set in H contains atleast two vertices. Also, some theorems related to the concept of Domination in Edge product Cluster Hypergraphs have been discussed and demonstrated here.

Keywords: cluster hypergraph, edge product cluster hypergraph, unit edge product cluster hypergraph

AMS Subject Classification: 05C65.

Introduction

The notation of sum graph was introduced by Harary F[1]. Thavamani first introduced the product analogue of sum graphs in 2011 also he introduced the edge product graphs and the edge product number of a graphs [1], [2].

Hypergraph generalizes the graphs where each edges of the hypergraph contains more than two vertices. Berge introduced the concept of Hypergraph in 1973 [3],[4]. The notation of domination in Hypergraphs was studied by B.D Acharya in 2007. In 2021, Kishor F Pawar and Megha M Jadhav introduced the concept of Domination in Edge Product Hypergraphs [5].

S. Samantha introduced the concept of Cluster Hypergraphs in 2020. To generalize the hypergraph concept cluster hypergraphs were introduced in which cluster nodes are permitted [6]. Mary Christal Flower .C and Befija Minnie .J together introduced the concept of Edge Product in Cluster Hypergraphs [7]. In this article the concept of Domination in Edge Product Cluster Hypergraphs is applied and also, the same concept has been extended to prove some theorems related to Domination in Edge Product Cluster Hypergraphs.

2 Preliminaries

Definition 2.1. Let X be a nonempty set and let V_X be a subset of $P(X)$ such that $V_X \neq \varphi$ and $X \subset V_X$. Now E be a multi-set whose elements belong to $P(P(X))$ such that

(i) $E \neq \varphi$

(ii) for each element $e \in E$, there exists atleast one element $v \in V_X$ such that $v \in e$.

Then $H = (V_X, E)$ is said to be *Cluster Hypergraph* where V_X is said to be a vertex set and E is said to be multi-hyper edge set [8].

Definition 2.2. Let $H = (V_X, E)$ be a simple cluster hypergraph with vertex set $V_X\ (H)$ and edge set $E(H)$. Let I be the set of positive integers such that $|E| = |I|$. Then any bijection $f: E \to I$ is called an Edge Function of the cluster hypergraph H [7].

Definition 2.3. The function $F(v) = \prod\{f(e);\ edge\ e\ incident\ to\ the\ vertex\ v\}$ on $V_X\ (H)$ is called an Edge product Function of the edge function f [7].

Definition 2.4. The cluster hypergraph $H = (V_X, E)$ is said to be an Edge Product Cluster Hypergraph if there exists an edge function $f: E \to I$ such that the edge function f and the corresponding edge product function F of f on $V_X(H)$ have the following two conditions:

(i) $F(\boldsymbol{v}) \in I$ for every $\boldsymbol{v} \in V_X(H)$

(ii) if $f(e_1) \times f(e_2) \times \ldots \times f(e_i) \in I$ for some edges $e_1, e_2, \ldots e_i \in E(H)$, then the edges $e_1, e_2, \ldots e_i$ are all incident to some vertex $v \in V_X(H)$ [7].

Definition 2.5. Let $H = (V_X, E)$ be an edge product cluster hypergraph. Then H is said to be a *Unit Edge Product Cluster Hypergraph* if there exists an edge function $f: E \to I$ such that $1 \in I$ [7].

3 Main Results

Definition 3.1. Let $H = (V_X, E)$ be an edge product cluster hypergraph. A Dominating Set of H is a set of vertices $S \subseteq V_X(H)$ such that for every vertex $x \in V_X(H) - S$ there exists an edge $e \in E(H)$ for which $x \in e$, $e \cap S \neq \varphi$. That is, every vertex $x \in V_X(H) - S$ is adjacent to vertex in S.

Example 3.1. Consider $H = (V_X, E)$ be an edge product cluster hypergraph where $V_X(H)$=$\{v_1, v_2, v_3 = \{v_1, v_2\}, v_4, v_5, \ldots, v_{24}\}$ and $E(H) = \{e_1, e_2, \ldots, e_{10}\}$. The edges of H is defined as $e_1 = \{v_1, v_3\}$, $e_2 = \{v_2, v_3\}$, $e_3 = \{v_3, v_4, v_5, \ldots, v_{10}, v_{11}\}$, $e_4 = \{v_3, v_4, v_5, v_{12}\}$, $e_5 = \{v_3, v_4, v_5, v_{13}, v_{14}\}$, $e_6 = \{v_6, v_7, v_{15}\}$, $e_7 = \{v_6, v_7, v_{16}, v_{17}, v_{18}\}$, $e_8 = \{v_6, v_7, v_{19}, v_{20}\}$, $e_9 = \{v_{21}, v_{22}\}$, $e_{10} = \{v_{23}, v_{24}\}$. Now the edge function $f: E \to I$ is determined by $f(e_1) = 7$, $f(e_2) = 13$, $f(e_3) = 11$, $f(e_4) = 30$, $f(e_5) = 4$, $f(e_6) = 3$, $f(e_7) = 20$, $f(e_8) = 2$, $f(e_9) = 120120$, $f(e_{10}) = 1320$. Then the edge product function F of f is determined by $F(v_1) = 7$, $F(v_2) = 13$, $F(v_3) = 120120$, $F(v_4) = F(v_5) = F(v_6) = F(v_7) = 1320$, $F(v_8) = F(v_9) = F(v_{10}) = F(v_{11}) = 11$, $F(v_{12}) = 30$, $F(v_{13}) = F(v_{14}) = 4$, $F(v_{15}) = 3$, $F(v_{16}) = F(v_{17}) = F(v_{18}) = 20$, $F(v_{19}) = F(v_{20}) = 2$, $F(v_{21}) = F(v_{22}) = 120120$, $F(v_{23}) = F(v_{24}) = 1320$..

Here v_3 dominates $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}\}$, v_6 dominates $\{v_{15}, v_{16}, v_{17}, v_{18}, v_{19}, v_{20}\}$, v_{21} dominates v_{22} and v_{23} dominates v_{24}.

Therefore, $S = \{v_3, v_6, v_{21}, v_{23}\}$ with cardinality 4 dominates H. Hence, $\gamma(H) = 4$.

Remark 3.2. For an edge product cluster hypergraph there can be more than one minimum dominating set in H. From example 3.1, $S_1 = \{v_3, v_6, v_{22}, v_{24}\}$, $S_2 = \{v_3, v_6, v_{22}, v_{23}\}$, $S_3 = \{v_3, v_6, v_{21}, v_{24}\}$, $S_4 = \{v_3, v_7, v_{21}, v_{23}\}$, $S_5 = \{v_3, v_7, v_{22}, v_{24}\}$, $S_6 = \{v_3, v_7, v_{22}, v_{23}\}$ and $S_7 = \{v_3, v_7, v_{21}, v_{24}\}$ are the minimum dominating sets in H .

Theorem 3.3. Suppose I indicates the set of positive integers with $1 \notin I$. Consider the edge product cluster hypergraph $H = (V_X, E)$. Then every dominating set in H contains atleast two vertices.

Proof. Consider the edge product cluster hypergraph $H = (V_X, E)$. If S is a dominating set of H. It is necessary to claim that S contains atleast two vertices. On the contrary, suppose $S = \{u\}$ be a dominating set in H. Since $1 \notin I$, theorem 3.1 in [7] implies that H is disconnected. This shows that H has atleast two components, say H_1, H_2. Then obviously $u \in H_1$ or $u \in H_2$. If $u \in H_1$, then every vertices in H_2 is not dominated by S. Similarly, If $u \in H_2$, then every vertices in H_1 is not dominated by S. From the both cases it can be obtained that S does not form a dominating set for H which contradicts the previous assertion. Thus, every dominating set in H contains atleast two vertices.

Theorem 3.4. If $H = (V_X, E)$ is a unit edge product cluster hypergraph with a unit edge e , then e serves as a dominating set for H .

Proof. If $H = (V_X, E)$ is a unit edge product cluster hypergraph with a unit edge. Then $f(e) = 1$. By theorem 3.3 in [7], each edges in H must be adjacent to e. Thus, the intersection of any edge together e is non-empty. For each $v \in V_X(H) - \{e\}$ there exists $u \in e$ in such a way that u dominates v. Hence, e serves as a dominating set for H.

Corollary 3.5. Consider the unit edge product cluster hypergraph $\mathrm{H} = (\mathrm{V_X}, \mathrm{E})$ with a unit edge e. Then the domination number of H is $\gamma(\mathrm{H}) \leq |\mathrm{e}|$.

Proof. Consider the unit edge product cluster hypergraph $\mathrm{H} = (\mathrm{V_X}, \mathrm{E})$ with a unit edge e. By theorem 3.4, e dominates H and so the domination number of H is $\gamma(\mathrm{H}) \leq |\mathrm{e}|$.

Theorem 3.6. Consider the unit edge product cluster hypergraph $\mathrm{H} = (\mathrm{V_X}, \mathrm{E})$ with a unit edge e. Then every dominating set of H contains either vertices from e or a pendent vertex in H.

Proof. Consider the unit edge product cluster hypergraph $\mathrm{H} = (\mathrm{V_X}, \mathrm{E})$ with a unit edge e. Let S be a dominating set of H. If $e \subseteq S$, then there is nothing to prove. Suppose $e \nsubseteq S$. Let m be any edge in H. By theorem 3.3 in [7], each edge in H must be adjacent to e. Therefore, for every vertex $v \in m$ there is a vertex $u \in S$ such that either $u \in e \cap m$ or u is a pendant vertex. Since $e \nsubseteq S$, so that $u \notin e$. Therefore, u must be a pendant vertex in H. Hence every dominating set of H contains either vertices from e or a pendant vertices in H.

Theorem 3.7. Suppose $\mathrm{H} = (\mathrm{V_X}, \mathrm{E})$ be a unit edge product cluster hypergraph with a unit edge. Then $\gamma(H) \leq |e| - k$, k represents the count of pendant vertices in e .

Proof. Suppose $\mathrm{H} = (\mathrm{V_X}, \mathrm{E})$ be a unit edge product cluster hypergraph with a unit edge e. Then $f(e) = 1$. By theorem 3.3 in [7], each edges in H must be adjacent to e. It implies that $\{u_1, u_2, \ldots, u_k\}$ be the collection of pendant vertices in e, $\{v_1, v_2, \ldots, v_l\}$ be the collection of non-pendant vertices in e. Therefore $v_i \in e \cap e_i$, the set $\{v_1, v_2, \ldots, v_l\}$ dominates all the remaining vertices of e_i for $i = 1, 2, \ldots, l$ and the pendant vertices $\{u_1, u_2, \ldots, u_k\}$. Moreover the vertices inside the cluster vertices are dominated by $\{v_1, v_2, \ldots, v_l\}$. Thus, $\{v_1, v_2, \ldots, v_l\}$ dominates all the vertices in H of cardinality $|e| - k$. Hence, $\gamma(H) \leq |\{v_1, v_2, \ldots, v_l\}| \leq |e| - k$.

Remark 3.8. The constraint in theorem 3.7 is exact. Consider the unit edge product cluster hypergraph H which is shown in figure 5.1.

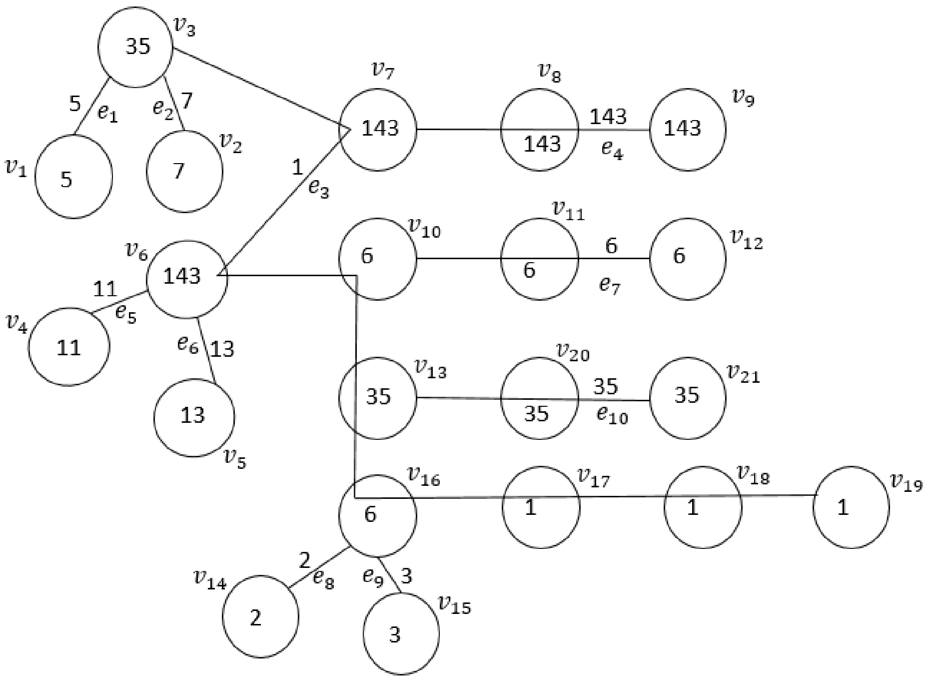

Figure 5.1

Here $S = \{v_3, v_6, v_7, v_{10}, v_{13}, v_{16}\}$ dominates H so that $\gamma(H) = 6$. Also, e_3 be the unit edge where $e_3 = \{v_3, v_6, v_7, v_{10}, v_{13}, v_{16}, v_{17}, v_{18}, v_{19}\}$. Clearly $|e_3| = 9$ and number of pendant vertices $k = 3$. Therefore, $|e_3| - k = 6 = \gamma(H)$.

The inequality in theorem 3.7 is strict. Consider the unit edge product cluster hypergraph H which is shown in figure 5.2.

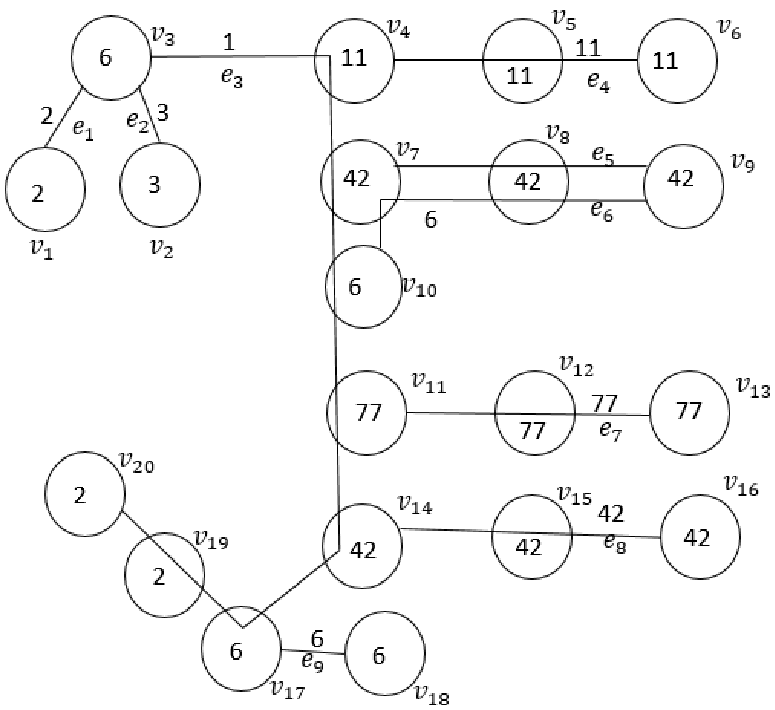

Figure 5.2

Here $S = \{v_3, v_4, v_{11}, v_{14}, v_{17}\}$ dominates H, so that $\gamma(H) = 5$. Also, e_3 be the unit edge and $e_3 = \{v_3, v_4, v_7, v_{11}, v_{14}, v_{17}, v_{19}, v_{20}\}$ is an unit edge of H. Clearly, $|e_3| = 8$ and number of pendant vertices $k = 2$. Therefore, $|e_3| - k =$

$6 > \gamma(H)$ Which implies that $\gamma(H) < |e_3| - k$.

Theorem 3.9. Consider the unit edge product cluster hypergraph $\mathrm{H} = (\mathrm{V_X}, \mathrm{E})$ with a unit edge e containing k pendant vertices. Suppose $e_1, e_2, \ldots, e_{m-1}$ be the non-unit edges in H. If $e_i \cap e_j = \varphi$ in $H - e$ for all $1 \leq i \neq j \leq m-1$, then $\gamma(H) = |e| - k$.

Proof. Consider the unit edge product cluster hypergraph $\mathrm{H} = (\mathrm{V_X}, \mathrm{E})$ with a unit edge e containing k pendant vertices.

Suppose $e_1, e_2, \ldots, e_{m-1}$ be the non-unit edges in H. Assume that $e_i \cap e_j = \varphi$ for $i \neq j$. Since $f(e) = 1$, by theorem 3.3, in [7] each edges in H must be adjacent to e. Therefore, $e \cap e_i \neq \varphi$ for $i = 1, 2, \ldots, m-1$.

Let $v_i \in e \cap e_i$, $i = 1, 2, \ldots, m-1$. Since $e_i \cap e_j = \varphi$, v_i be the only vertex common to edges e and e_i, $i = 1, 2, \ldots, m-1$, and also v_i is the only non-pendant vertex of e_i. Therefore, it can be easily observed that the edges $e_1, e_2, \ldots, e_{m-1}$ are the pendant edges in H. So there should be atleast $v_1, v_2, \ldots, v_{m-1}$ vertices to dominate the vertices contained in the edges $e_1, e_2, \ldots, e_{m-1}$. Also, the pendant vertices in e are also dominated by the non-pendant vertices $v_1, v_2, \ldots, v_{m-1}$.

Therefore, it is necessary to have atleast $|e| - k$ number of vertices to dominate the unit edge product cluster hypergraph H. So $\gamma(H) \geq |e| - k$. By theorem 3.7, it is conclude that $\gamma(H) = |e| - k$.

Remark 3.10. The reverse implication of the above theorem may not hold true always,

For example, consider $\mathrm{H} = (\mathrm{V_X}, \mathrm{E})$ where $\mathrm{V_X}(H) = \{v_1, v_2, v_3, v_4 = \{v_1, v_2, v_3\}, v_5, \ldots, v_{23}\}$ and $E(H) = \{e_1, e_2, \ldots, e_{10}\}$. The edges of H is defined as $e_1 = \{v_4, v_5, v_8, v_{12}, v_{15}, v_{18}, v_{21}, v_{22}, v_{23}\}$, $e_2 = \{v_1, v_4\}$, $e_3 = \{v_2, v_4\}$, $e_4 = \{v_3, v_4\}$, $e_5 = \{v_5, v_6, v_7\}$, $e_6 = \{v_5, v_9, v_{10}\}$, $e_7 = \{v_8, v_9, v_{10}, v_{11}\}$, $e_8 = \{v_{12}, v_{13}, v_{14}\}$, $e_9 = \{v_{15}, v_{16}, v_{17}\}$, $e_{10} = \{v_{18}, v_{19}, v_{20}\}$. Now the edge function

$f: E \to I$ is determined by $f(e_1) = 1$, $f(e_2) = 17$, $f(e_3) = 3$, $f(e_4) = 11$, $f(e_5) = 13$, $f(e_6) = 5$, $f(e_7) = 30$, $f(e_8) = 150$, $f(e_9) = 561$, $f(e_{10}) = 65$. Then the edge product function F of f is determined by $F(v_1) = 17$, $F(v_2) = 3$, $F(v_3) = 11$, $F(v_4) = 561$, $F(v_5) = 65$, $F(v_6) = F(v_7) = 13$, $F(v_8) = 30$, $F(v_9) = F(v_{10}) = 150$, $F(v_{11}) = 30$, $F(v_{12}) = F(v_{13}) = F(v_{14}) = 150$, $F(v_{15}) =$F(v_{16})=$F(v_{17}) = 561$, $F(v_{18}) = F(v_{19}) = F(v_{20}) = 65$, $F(v_{21}) = F(v_{22}) = F(v_{23}) = 1$.

Here $S = \{v_4, v_5, v_8, v_{12}, v_{15}, v_{18}\}$ dominates H. Therefore, $\gamma(H) = |S| = |e| - 3 = |e| - k$. Thus $\gamma(H) = |e| - k$. But the edges e_5 and e_6 are adjacent with the unit edge e_1. Also, vertices v_5, v_9, v_{10} are contained in both the edges e_5 and e_6 and so $e_5 \cap e_6 \neq \varphi$.

In H the unit edge e_1 contains 3 pendant vertices with $\gamma(H) = |e| - 3$. But $e_5 \cap e_6 \neq \varphi$ non-unit edges e_5 and e_6.

Theorem 3.11. Consider the unit edge product cluster hypergraph $\mathrm{H} = (\mathrm{V_X}, \mathrm{E})$ with a unit edge e. Then any two distinct non-unit edges are non-adjacent if and only if $\gamma(H) = m - 1$.

Proof. Consider the unit edge product cluster hypergraph $\mathrm{H} = (\mathrm{V_X}, \mathrm{E})$ with a unit edge e. Assume that any two distinct non-unit edges are non-adjacent. To prove that $\gamma(H) = m - 1$. Let e_1 and e_2 be any two distinct non-unit edges in H. Since the e_1 and e_2 are non-adjacent and their intersection is empty. To dominate the vertices in e_1, choose either a vertex in e_1 or a vertex in e incident with e_1. Similarly, to dominate the vertices in e_2, choose either a vertex in e_2 or a vertex in e incident with e_2. This is true for all non-unit edges $e_1, e_2, \ldots, e_{m-1}$ of H. Therefore, there are atleast $m - 1$ vertices to dominate H. Hence $\gamma(H) \geq m - 1$. On the other hand, since H is a unit edge product cluster hypergraph, for every $e_i \in E$ there exists a vertex $x_i \in e \cap e_i$, $1 \leq i \leq m - 1$. Thus the set S containing the vertices $x_1, x_2, \ldots, x_{m-1}$ forms a dominating set in H and so $\gamma(H) \leq |S| = m - 1$. Hence $\gamma(H) = m - 1$.

Conversely, assume that $\gamma(H) = m-1$. Let $e_1, e_2, \ldots, e_{m-1}$ be the non-unit edges in H . To prove that any two distinct non-unit edges are non-adjacent. That is to prove that $e_i \cap e_j = \varphi,\ 1 \le i \ne j \le m-1$. On contrary suppose there exists edges e_1 and e_2 where e_1 and $e_2$is adjacent. Then, the intersection of e_1 and e_2 is non-empty. Let $x \in e_1 \cap e_2 \cap e$. This is possible because H is a unit edge product cluster hypergraph. Also, the unit edge e is adjacent with all the non-unit edges in H. Thus $x_i \in e \cap e_i,\ 1 \le i \le m-1$. Therefore, selecting $S = \{x_3, x_4, \ldots, x_{m-1}\} \cup \{x\}$ forms a dominating set of H and so $\gamma(H) \le |S| = m-3+1 = m-2 < m-1$. Therefore, $\gamma(H) < m-1$ which contradicts the previous assertion. So our assumption is wrong. Hence any two distinct non-unit edges are non-adjacent.

4 Conclusion

In this article the concept Domination in Edge Product Cluster Hypergraph has been applied and also, the same concept is extended to prove some theorems related to the Domination Number in Edge Product Cluster Hypergraph. In future studies various parameters of domination and domination number in Edge Product Cluster Hypergraphs can be determined for some other cluster hypergraphs.

References

[1] J.P. Thavamani and D.S.T Ramesh, *Edge product number of graphs in paths,* Journal of Mathematical Sciences and Computer Applications, 92011),78-84.

[2] J.P. Thavamani and D.S.T Ramesh, *Edge product number of graphs and its properties,* The IUP Journal of Computational Mathematics, 4 (2011), 30-38.

[3] Berge.C, *Graphs and Hypergraph*, Elsevier, Amsterdam, The Netherlands, (1973).

[4] Berge.C, *Hypergraph*, North-Holland, Amsterdam, The Netherlands, (1989).

[5] kishor F. Pawar and Megha M. Jadhav *On Domination In an Edge*

Product Cluster Hypergraphs, Journal of Hyperstructures, 10 (2) (2021), 108-117, ISSN : 2322- 1666 print / 2251-8436 online.

[6] Megha M. Jadhav and Kishor F. Pawar, *On Edge Product Number of Hyper-graph*, Palastine Journal Of Mathematics, Vol. 11 (4) (2022), 183-194.

[7] Befija Minnie. J and Mary Christal Flower. C, *Edge Product In Cluster Hyper- graphs*, European Chemical Bulletin, 12, (2023), 7732-7738.

[8] Samanta S, Lee JG, Naseem U, Khan SK, Das K, *Concept of Coloring of Cluster Hypergraphs*. Math Probl Eng (2020). https://doi.org/10.1155/2020/3705156.

[9] Berge.C, *Theory of Graphs and its Applications*, Methuen, London (1962).

[10] Harary F, *Sum graphs over all the integers*, Discrete Math, 124: 99-105 (1994).

[11] Harary F, *Sum Graphs and difference graphs*, Congress Number 72: 101-108 (1990).

[12] J.P. Thavamani and D.S.T Ramesh, *Cyclic Graphs-Edge Product Number*, International Journal Sciences, Environment and Technology, Vol.1, No 5, 2012, 454-461.

[13] Maity A, Samanta S, Mondal S, Dubey V, *A Study of Cluster Hypergraphs and Its Properties*, Social Network Analysis and Minning, (2021).

[14] Megha M. Jadhav and Kishor F. Pawar, On Edge Product Hypergraphs, Journal of Hyperstructures, 10(1) (2021), 1-12, ISSN: 2322-1666/2251-8436 online.

[15] Ore .O, Theory of Graphs, Amer. Maths. Soc. Collq. Publ., 38 (Amer. Math. Soc., Providence, RI), (1962).

[16] Samanta S, Muhiudin G, Alanazi AM, Das K, A Mathematical approach on representations of Competition: Competition Cluster Hypergraphs. Mat prob Eng, (2020)