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A Revision of Relaxed Steepest Descent, Gradient Descent, Modified Ellipsoid, David-Fletcher-Powell Variable Metric, Newton's, and Fletcher-Reeves Conjugate Methods From the Dynamics on an Invariant Manifold: A Journey of Mathematical Optimization

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Abstract

The goal of optimization theory is to minimize an objective function while taking a set of constraints into account. The design, management, operation, and analysis of systems in the actual world depend heavily on this field. For many decades, there has been a vigorous research focus on the creation of effective minimization strategies and numerical algorithms. Finding the steepest decent method's computational comparison rate is the primary goal of this study. Gradient Descent is one of the most popular methods to pick the model that best fits the training data i.e. the model that minimizes the loss function for example, minimizing the residual sum of squares in linear regression. Stochastic Gradient Descent is a stochastic, as in probabilistic, Spin on gradient descent. It improves on the limitations of gradient descent method and performance much better in large scale datasets. In this work, we study and compare all i.e. modified ellipsoid method, David-Fletcher-Powell variable metric method, Newton's method, and Fletcher-Reeves conjugate gradient techniquesin optimization theory.

Keywords:Unconstrained convex minimization, constrained optimization, design, control, etc.

Introduction

An optimization issue is one where a collection of workable solutions is presented, and the user must choose an optional answer in one or more sense. There are many different types of optimization challenges. The objective, which includes inequalities or equalities, free or non-negative variables, and the mathematical characteristics of the functions involved in the objective or restrictions, may all change. One of the main quantitative methods in operational research is mathematical programming. The use of optimization methods in management economic, industrial, and other processes is known as mathematical programming. It works especially well at the operational level to solve recurring operational issues.

This may be explained analytically as the optimization of a numerical function that characterizes the different activity levels in the presence of several factors. This function is accompanied with certain limitations. When handling situations where the decision maker must distribute scarce or restricted resources to achieve the decision maker's objectives while reaching the maximum degree of quantifiable goals or objectives, a mathematical programming method is helpful. There may or may not be a huge number of solutions, depending on the presumptions and unique features of a specific situation. These might be unlimited or finite.Among the range of practical. The optimal solution satisfies the specified criteria to the greatest extent possible (see [10-13]).Formulated the general linear programming problem and developed the simplex method for its solution (1963). Early applications were limited to military operations. Since that time, several useful extensions of the basic linear programming model have been developed. Any situation in which a choice among available alternatives must be considered defines a decision problem. Such decisions are made to influence future events. The decision maker has to make certain assumptions regarding the certainty or uncertainty of own estimation of future states of nature. Problems involving certainty are called deterministic approach seems to be prevailing in the investigation of mathematical programming and its application. However, this approach can often encounter stumbling blocks and even an elaborate and sophisticated deterministic programming model may have to be discarded because the data it requires may not exist or may exist but be of such poor quality that the results obtained from it could not be Relied upon. As a result, these difficulties arise because the operation which we are performing to stimulate mathematically is performed in the presence of uncertainties due to the occurrence of unpredictable events. Randomness sources may be many, depending on the nature and type of operation under study. For instance, in financial planning, decisions must be taken before the variables like demands, available capacities, prices and interest rates etc. Are known and as such must be treated as random variables. In the design of mechanical system, the actual dimensions of any machined part has to be taken as a random variable since the dimension may lie within a specified tolerance band. Another example is the designing of aircraft and rockets in which the actual load acting on the variable is unpredictable and hence random. This is dependent on the atmosphere conditions at the time of the flight, which cannot be predicted precisely in advance.

To ensure a certain class of reliability for the solutions to optimization problems containing random data, it has become an accepted approach to introduce probabilistic constraints into the model. In fact deterministic development is based on assumptions constantly violated by random factors. For the classical deterministic approach to be more realistic, the assumption of absolute knowledge of the data required must be relaxed, and the effects of stochastic uncertainties have to be taken into consideration in addition to the assumptions that the required data is completely known.

Probabilistic problems, however, occur when the decision maker does not assume certainty about the outcome of the course of action. Uncertainty can arise in many ways. It is possible for the outcome of a particular action to be influenced by some chance event. There are times when the distribution of chance events is known or partially understood. There are times when uncertainty arises as a result of competitors or enemies. Madansky [14] pointed out that the area of programming under uncertainty cannot be useful stated as a single problem, Dantzing [10,12], Ferguson and Dantzig [15]. The situations of decision makers facing random parameters in optimization can be found in Sengupta [16], Vajda [13], Kall [17], Kolbin [18], Kall and Prekopa [19], Dempester [20], Ermoliev and Wets [21], Frauendorfer [22], etc.

Stochastic uncertainty influences a programming model in two ways. Firstly, there is a direct effect resulting from random phenomena whose probability distributions of anticipation are clearly known. Specifying the probability distributions of a random phenomenon has a direct effect. Tinter [23] distinguishes the two as subjective risk and subject uncertainty. A technical distinction is, therefore, sometimes made between risk and uncertainty to indicate that the probability distributions of random variables involved are unknown respectively. The former field leads to the stochastic or probabilistic programming. Stochastic programming problems are characterized by their difficulty of solutions. As soon as one or more of the parameters of the problem become random variables, even the simplest linear problems can and often do become non-linear. One basic difficulty is that such a problem is capable of many formulations with only fragmentary results for each formulation, Madansky [24].Mathematical programming is a branch of optimization theory in which one determines the largest or smallest value of a function of several variables. This is subject possibly to one or more constraints. Mathematical programming is effective in solving problems in which the decision maker must allocate scarce or limited resources to achieve the highest level of measurable goals or objectives.Charnes and Cooper [25] proposed the chance constrained programming procedure for solving linear stochastic programming problem. Chance constraints are transformed into nonlinear constraints of deterministic nonlinearity that have a normal distribution and are distributed independently of each other as nonlinearities that are distributed normally. Ecker and Kuptershmid [26] provide a computational evidence that the ellipsoid algorithm is extremely robust and relative to efficiency. For a complete study of ellipsoid algorithm see the survey of ellipsoid algorithm given by Bland et al.,[27]. A linear approximation for chance constrained programming was given by Olson and Scott [28]. A piecewise linear goal programming method has been used by Rakes et al. [29] in order to solve models with chance constraints. Weintrub and Vera [30] considered the constraints (1) of the problem defined in (1)-(2) of the coming part for $\geq$ case taking $\propto_{lk}$ as a randomvariables distributed normally and solve it by cutting plane algorithm. In this note, we consider the constraints (2) for $\leq$ case taking both $\propto_{lk}$ and $\beta_l$as random variables and solving such problems by using ellipsoid algorithm.

The rate of convergence of the steepest-decent method is the best linear even for a quadratic cost function. It is possible to accelerate this rate of convergenve of the steepest-decent method if the condition number of Hessian of the cost function can be reduced by scaling the design variables(see [1-4]). The first non-linear conjugate gradient method was introduced by Fletcher and Reeves, it is one of the earliest known techniques for solving non-linear optimization problems (see [5-6]).

The David on- Fletcher-Powell penalty function method is a technique that been used successively to solve constrained minimization problems. The method was devised by combining an exterior penalty function with a performance function for solving constrained minimization problems (see [7]).

In 1979, the Russian mathematician L.G. Khachiyan published his famous paper with the title "A polynomial Algorithm in Linear programming (see [8]). He was able to show that LP can be solved efficiently; more precisely that LP belongs to the class of polynomially solvable problems. Khachiyan's approach was based on ideas similar to ellipsoid method arising from convex optimization. These methods were developed by D. Yuddin and A. Nemirovski (see [9]). Ellipsoid method solves the problem of finding a feasiblepoint of a system of linear inequalities. This problem is closely related to the problem of solving the LPP

$$\begin{aligned} &\max \beta^T \alpha \\ &\alpha \\ &s.t. \gamma^T \alpha \leq \curlywedge, \\ &\alpha \geq 0 \end{aligned}$$

Comparison of Unconstrained convex minimization methods

In an unconstrained minimization problem, if the first and second derivatives of the objective can be evaluated easily (either in closed form or by a finite difference scheme), and if the number of design variables is not large ($n \geq 50$), one of the quasi-Newton methods is being used effectively. For n greater than about 50, the storage and inversion of the Hessian marix on computer at each stage becomes quite tedious and the variable matrix method is considered more useful. As the problem size increases (beyond n=100 or so), the conjugate gradient method is found more powerful.

In many practical problems, the first derivatives of the function can be computed more accurately than the second derivatives. In such cases, the variable metric method becomes an obvious choice of minimization up to a value of n=100. If the evaluation of the derivatives of the function is extremely difficult or if the function does not possess continuous derivatives, the Powell's method is used to solve the problem efficiently.

The Ellipsoid method requires only the first derivative of the functions involved. Further with regard to the time required for developing the computer program and to the accuracy of the solution, the Ellipsoid method is found to be the robust one. The Ellipsoid method however, being able to handle many of the smooth non-convexities, it converge to the solution for larger set of problems.

Numerical Example:

We consider the minimization of the following function for comparing the relative efficiencies of the various unconstrained methods.

$$\text{Min } \sigma(\theta) = 100({\theta_1}^2 - \theta_2)^2 + (1 - \theta_1)^2$$

The points reached at various iterations by different methods and the respective values of the function are listed in the following tables.

(i) Steepest Descent Method.

Iteration	θ_1	θ_2	$\sigma(\theta)$
0	-0.100	1.000	4.00
1	-0.995	1.000	3.99
2	-0.995	0.990	3.98
3	-0.990	0.990	3.97
4	-0.990	0.979	3.96
5	-0.984	0.979	3.95
.	... ...	... ...	... ...
.	... ...	... ...	... ...
.	... ...	... ...	... ...
70	-0.742	0.550	3.03
80	-0.698	0.487	2.88
90	-0.950	0.422	2.72
100	-0.594	0.352	2.54

(ii) Fletcher-Reeves Conjugate Gradient Method.

Iteration	θ_1	θ_2	$\sigma(\theta)$
1	-0.1000	1.000	4.00
2	-0.983	0.988	3.99

3	−0.472	0.248	3.98
4	−0.451	0.294	2.23
5	−0.358	0.100	2.11
.	… …	… …	… …
.	… …	… …	… …
.	… …	… …	… …
.	… …	… …	… …
70	1.0870	1.173	1.02
71	1.0870	1.012	1.01
72	1.0070	1.009	0.00
73	1.0040	1.009	0.00

(iii) Newton's Method.

Iteration	θ_1	θ_2	$\sigma(\theta)$
0	−0.1000	1.000	4.00
1	1.000	−3.000	1599.00
2	1.000	0.999	0.00
3	1.000	1.000	0.00

(iv) David –Fletcher-Powell Variable Metric method.

Iteration	θ_1	θ_2	$\sigma(\theta)$
0	−0.1000	1.000	4.00
1	−0.995	1.000	4.00
2	−0.775	0.562	3.99
4	−0.254	0.029	3.31
.	… …	… …	… …
.	… …	… …	… …
.	… …	… …	… …
17	0.961	0.927	0.01
19	1.000	1.000	0.00

(v) Modified Ellipsoid Method

Iteration	θ_1	θ_2	$\sigma(\theta)$
0	-0.1000	1.000	4.000
1	0.000	1.000	101.000
2	0.004	-0.154	3.380
3	0.009	0.615	1.880
4	0.057	0.103	… …
.	… …	… …	… …
.	… …	… …	… …
.	… …	… …	… …
.	… …	… …	… …
25	0.882	0.784	0.017
26	0.993	0.853	0.047
27	1.000	1.000	0.000

In the above numerical example it was found that the convergence of the Steepest Descent Method was very slow and after 100 iterations, it reached a point where the Fletcher-Reeves conjugate gradient method reached in only 4 iterations. The convergence of the Newton method is seen to be extremely rapid although the function value increased in the first iteration. The Ellipsoid Method started converging towards the optimal solution after 5 iterations.

Constrained Optimization (Convex case)

In convex programming problems involving explicit (non-linear expression for objective function, constraints with small or moderate number of variables the penalty function methods have been expected to work most efficiently. Out of these, the interior penalty function method is less efficient since even a feasible starting point leads to an infeasible point at the end of the minimization procedure. As the sequence of optimal points $\theta_1{}^*, \theta_2{}^* \ldots$ lies in the feasible region, and approaches the optimum point and feasibility simultaneously, this method is useful only when a starting feasible point cannot be found. If all constraints of the Optimization problem are linear, the gradient projection methods has been used as the best one. If the problem involves objective function and constraints that are implicitly dependent on the design vector (i.e. an analysis is to be needed to evaluate $\sigma_i(\theta), (i = 0, \ldots, n)$, the derivatives of the functions$\sigma_i(\theta)$ cannot be obtained in closed form. When these derivatives can be obtained by finite-difference formulae, the Zoutendijk's method of feasible direction has been used and is more efficient than the penalty function methods. However, if one intends to use approximation in evaluating $\sigma_i(\theta)$itself, the penalty function methods appear to be more promising. If the evaluation of $\sigma_i(\theta)$is extremely difficult and if one is Interested in finding only a near-optimal solution, the interior penalty function method was the obvious choice.

We consider the following example to have an idea about the comparative efficiencies of the above methods.

$$\text{Minimize}\sigma_i(\theta) = -15\theta_1 - 27\theta_2 - 36\theta_3 - 18\theta_4 - 12\theta_5$$
$$+30\theta_1{}^2 + 39\theta_2{}^2 + 10\theta_3{}^2 + 39\theta_4{}^2 + \theta^2$$
$$+40\theta_1\theta_2 - 62\theta_2\theta_4 + 64\theta_2\theta_5 - 12\theta_3\theta_4 - 20\theta_3\theta_5 - 40\theta_4\theta_5$$
$$+4\theta_1{}^3 + 8\theta_2{}^3\theta^3 + 6\theta_4{}^3 + 2\theta_5{}^3$$

subject to

$$16\theta_1 - 2\theta_2 - \theta_4 \leq 40$$
$$2\theta_2 - 0.4\theta_4 - 2\theta_5 \leq 2$$
$$\frac{7}{2}\theta_1 - 2\theta_3 \leq \frac{1}{4}$$
$$\theta_1 + -4\theta_4 - \theta_5 \leq 4$$
$$9\theta_2 + 2\theta_3 - \theta + 0.8\theta_5 \leq 4$$
$$-2\theta_1 - 4\theta_3 \leq 1$$
$$\theta_1 + \theta_2 + \theta_3 + \theta_4 + \theta_5 \leq 40$$
$$\theta_1 + 2\theta_2 + 3\theta_3 + 2\theta_4 + \theta_5 \leq 40$$
$$-\theta_1 - 2\theta_2 - 3\theta_3 - 2\theta_4 - 5\theta_5 \leq 60$$
$$-\theta - \theta_2 - \theta_3 - \theta - \theta_5 \leq -1$$
$$-\theta_i \leq 0, i = 1, \ldots, 5.$$

Different solution methods of convex programming have been implemented to solve the above problem and the results obtained have been shown in the tables given below. For all the methods the starting point is taken as $(0, 0, 0, 0, 1)$ with $\sigma(\theta) = 20$.

(i) Zoutendi jk's Method of Feasible Direction

Iteration	$\sigma(\theta)$	θ_1	θ_2	θ	θ_4	θ_5
2	−19.184	0.1337	0.0224	0.3169	0.5422	0.3403
5	−23.916	0.3001	0.3328	0.4000	0.6725	0.0286
10	−31.604	0.2568	0.3220	0.4892	0.4892	0.2978
20	−31.779	0.2738	0.2474	0.5143	0.5143	0.2191
45	−32.285	0.2983	0.3234	0.4521	0.4521	0.2645
100	−32.3486	0.3000	0.3327	0.4268	0.4268	0.2255

(ii) Rosen's Gradient Projection Method.

Iteration	$\sigma(\theta)$	θ_1	θ_2	θ_3	θ_4	θ_5
7	11.215	0.0302	0.000	0.0485	0.0500	0.9526
10	−24.363	0.2534	0.000	0.3767	0.4062	0.6159
15	−24.825	0.2534	0.000	0.3866	0.4063	0.6160
20	−31.582	0.3000	0.2735	0.3932	0.3887	0.4024
26	−32.043	0.3000	0.3556	0.4000	0.4924	0.1719
31	−32.329	0.3000	0.3403	0.4000	0.4449	0.2081
45	−32.348	0.3000	0.3341	0.4000	0.4299	0.2226

(iii) Interior Penalty Function Method

Value of penalty parameter	$\sigma(\theta)$	θ_1	θ_2	θ_3	θ_4	θ_5
1	−26.25	0.1762	0.2575	0.2868	0.5698	0.4272
2×10^{-2}	−31.23	0.2684	0.3208	0.3743	0.4675	0.2673
4×10^{-2}	−32.19	0.2954	0.3311	0.3961	0.4952	0.2327
8×10^{-2}	−32.34	0.2993	0.3332	0.3994	0.4292	0.2254

(iv) Modified Ellipsoid Method

Iteration	$\sigma(\theta)$	θ_1	θ_2	θ_3	θ	θ_5
4	−10.540	−0.302	1.000	1.740	1.000	1.000
5	19.987	0.309	−0.139	0.354	1.239	0.955
6	10.840	0.164	−0.146	0.490	1.302	0.905
9	−14.340	0.008	0.326	0.060	0.442	0.441
47	−26.976	0.239	0.295	0.313	0.570	0.590
65	−32.266	0.248	0.394	0.393	0.572	0.143

The Zoutendijk's method converged after 100 iterations with 20 evaluations of $\nabla\sigma_i$ and 20 evaluations of the constraint set Whereas Rosen's method converged after 45 iterations with 70 evaluations θ_i and $\nabla\sigma_i$. The Interior penalty function method took 315 evaluations of $\sigma_i(\theta)$ and 18 evaluations of second partial derivatives. The Ellipsoid Method started Converging after 9 iterations and give the result in 65 iterations.

Constrained optimization (Non-convex case)

In this section we mainly consider two situations:

(i) The case of minimizing a concave objective function in a convex feasible solution space; and

(ii) The case of minimizing a concave objective function in a non-convex feasible set.

The methods of Tui [1964], Ritter [1964], Hoffman [1981], Benson [1986] and Khan et al [1986] etc. come in the first situation. Tui [1964] developed a method which introduces a cut and reduces the feasible region. But the introduction of a cut always increased the number of extreme points of the feasible region. Later many modifications for improvement in the algorithm were made by Ritter [1964], Hoffman (1981) and Benson [1986] etc. Zwart [1973] demonstrated, the cycling of the method of Tui [1964] and Ritter [1964]. Khan (1986] developed a cutting plane method which decreased the number of extreme points since the cut passes through the second adjacent extreme points of the current solution.

Below we solve a numerical example using some of the above methods and then by the modified Ellipsoid Method. Here the objective function is concave and the constraint set is convex.

Numerical Example:

Minimize $\sigma(\theta) = 2{\theta_1}^2 - \theta_1\theta_2 - 2\theta_2$

Subject to

$\theta_1 + \theta_2 \leq 1$

$1.5\theta + \theta_2 \leq 1.4$

$-\theta_1 \leq 0$

$\theta \leq 10$

(i) Ritter's Method

Iteration	θ_1	θ_2	$\sigma(\theta)$
2	0	$-1/2$	-1
3	0	1	-2
4	0	$1/2$	-1
5	0	1	-2
.	.	.	.
.	.	.	.
.	.	.	.

(ii) Adjacent points Cutting Plane Method

Iteration	θ_1	θ_2	$\sigma(\theta)$
1	0	11	-2
2	7.6	-10	-19.52

(iii) Modified Ellipsoid Method

Iteration	θ_1	θ_2	$\sigma(\theta)$
0	0.000	0.000	0.0
1	0.000	7.542	−15.08
3	−0.431	2.658	−04.54
−	..	..	..
..	..	..	..
..	..	..	..
50	7.590	−09.995	−19.36
52	7.589	−09.986	−19.44
55	7.594	−09.992	−19.47
58	7.596	−09.995	−19.49
60	9.603	−10.006	−19.52

In the solution of the above numerical example it was found that in the Ritter's Method, the solution procedure becomes interminable, as long as we choose the next extreme point. The adjacent points cutting plane method converged in two steps. The Ellipsoid Method took about 60 iterations to converge the optimal solution.

As far as the situation of minimizing a concave objective function in a non-convex solution space is concerned, no method has got successful convergence other than Modified Ellipsoid Method. The Ellipsoid Method converged in a finite number of iterations in the problems with concave functions in the constrained set.

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