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A STUDY ON CONTINUOUS MODEL OF A THREE SPECIES COMPETITIVE ECOLOGY

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Abstract:
The focus of the paper is on a continuous model of a resource-constrained, three-species competitive ecology. The effort to outlive the other competitors is referred to as competition. There is competition between species for survival when they attempt to coexist in the same community, with the same resources, and in the same environment. When resources are scarce, a struggle for survival would ensue. Three simultaneous non-linear first ODEs make up the mathematical model equations. The model's eight equilibrium points are all found. If every eigen value in a complex system is negative, or if the values are real and have negative real parts, the system would be stable. The trajectories over the equilibrium locations are depicted. And next, there has been established system's global stability with the use of a suitably designed Lyapunov's function. Further, the Runge-Kutta fourth order technique is used to compute the numerical solutions for the growth rate equations.

Key words: Characteristic Equation, Competition, Equilibrium Point, Stable, Unstable.

1. Introduction:

Ecology is the study of how living things interact with their surroundings. Plants and animals are considered creatures, while the surroundings of animals are considered part of the environment. The study of living things (plants and animals) in connection to their environments and habits is known as ecology. This discipline of knowledge is a branch of evolutionary biology purported to explain how or to what extent the living beings are regulated in nature. Allied to the problem of population regulation is the problem of species distribution-prey-predator, competition and so on. The subject of ecology can be broadly sub-divided as auto-ecology (the study of single species populations) and synecology (the study of two or more communities). Synecological studies lead to the concept of the eco-system. This concept is a direct outcome of the intensive work of several life scientists/biologists and botanists of many generations. An eco-system may be considered as a unit that includes animals, plants and the physical environment in which these live. This area of knowledge seeks to explain how many different kinds of plants and animals can live together in the same place for many generations. Animals and plants share the same habitat. Sometimes they can only share for so long before some locally go extinct, but there are other circumstances when many different kinds persist in a habitat indefinitely. As such, ecology may also be referred as the study of distribution and abundance of species under habitat availing the same resources. The Ecological interactions can be broadly classified as Ammensalism, Competition, Commensalism, Neutralism, Mutualism, Predation and so on. The predator-prey equations, commonly known as the Lotka-Volterra equations[1,2], describe the interactions between two predator-prey or more species in an ecosystem. The Lotka-Volterra equations are a pair of first-order, nonlinear differential equations that describe the dynamics of biological systems or changes in population. These formulas are utilised in the literature on biology, ecology, and the environment, as well as in other disciplines like atmospheric chemistry, urban growth studies, economics, and tourism. Numerous ecologists made contributions to the expansion of this field of study[3, 4]. Numerous researchers have been drawn to study predator-prey systems, both continuous and discrete, because of their ability to display complicated dynamical behaviour [5-7]. Moreover, the evolution of contemporary applied statistics revolves around ecological applications [8–10].

One significant example of an interdisciplinary endeavour is mathematical modelling, which is the mathematical analysis of certain facets of many disciplines. Among these fields are biology, epidemiology, physiology, ecology, sociology, immunology, bio-economics, genetics, and pharmacokinetics. It has offered its exemplary services for the advancement of engineering in particular and science and technology in general ever since it was founded. It now serves as the foundation for advancements in modern science. Its scope of influence has several dimensions, and each field of mathematics has a distinct significance. In recent years, mathematical modelling has reached its pinnacle, permeated every aspect of society, and captured the interest of all. Numerous scholars [11–13] and mathematicians [14–17] have contributed significant viewpoints, worthwhile tenets, and intriguing applications to the field of mathematical ecology that allow for the analysis of the behaviours of various ecological models. In intercultural aspects, Hari Prasad investigated the local and global stability of numerous mathematical models (both discrete and continuous) of syn-ecology [17–23].

2. Basic Equations of the Model:

The set of non-linear first order ordinary differential equations that follows, using the following symbols, provides the model equations for the rivalry between the three species.

Notation Adopted:

$N_i(t)$: $S_i's$ population strength at time t, i =1, 2 , 3.

t : Time period.

a_i : $S_i's$ natural birth rate, i =1,2,3 .

a_{ii} : $S_i's$ self- inhibition coefficients, i =1,2,3.

a_{12}, a_{21} : Interaction coefficients of S_1 due to S_2 and S_2 due to S_1.

a_{13}, a_{31} : Interaction coefficients of S_1 due to S_3 and S_3 due to S_1.

$k_i = \frac{a_i}{a_{ii}}$: Carrying capacity of S_i, i =1,2,3 .

Additionally, it is expected that the model constants $a_1, a_2, a_3, a_{11}, a_{22}, a_{33}, k_1, k_2, k_3$, a_{12}, a_{21}, a_{13}, a_{31} are either zero or positive, and that the variables, N_1, N_2 & N_3, are either zero or positive.

Governing equations for the three species:

$$\frac{dN_1}{dt} = a_1N_1 - a_{11}{N_1}^2 - a_{12}N_1N_2 - a_{13}N_1N_3 \quad (1)$$

$$\frac{dN_2}{dt} = a_2N_2 - a_{22}{N_2}^2 - a_{21}N_1N_2 \quad (2)$$

$$\frac{dN_3}{dt} = a_3N_3 - a_{33}{N_3}^2 - a_{31}N_1N_3 \quad (3)$$

3. Equilibrium States:

After investigating the system has **Eight** equilibrium states given by

$$\text{for } i = 1,2,3, \quad \frac{dN_i}{dt} = 0 \quad (4)$$

And they are

$E_1 : \overline{N}_1 = 0, \overline{N}_2 = 0, \overline{N}_3 = 0$; E_2 : $\overline{N}_2 = 0, \overline{N}_1 = k_1, \overline{N}_3 = 0$

$E_3 : \overline{N}_1 = 0, \overline{N}_2 = k_2, \overline{N}_3 = 0$; $E_4 : \overline{N}_1 = 0, \overline{N}_2 = 0, \overline{N}_3 = k_3$

$E_5 : \overline{N}_1 = 0, \overline{N}_2 = k_2, \overline{N}_3 = k_3$; $E_6 : \overline{N}_1 = \frac{k_1 - \beta_{12}k_2}{1-\beta_{21}\beta_{12}}, \overline{N}_2 = \frac{k_2 - \beta_{21}k_1}{1-\beta_{21}\beta_{12}}, \overline{N}_3 = 0$,

E_6 would exist only when $k_1 > \beta_{12}k_2$, $k_2 > \beta_{21}k_1$ and $1 - \beta_{21}\beta_{12} > 0$

$E_7 : \overline{N}_1 = \frac{k_1 - \beta_{13}k_3}{1-\beta_{13}\beta_{31}}, \overline{N}_3 = \frac{k_3 - \beta_{31}k_1}{1-\beta_{13}\beta_{31}}, \overline{N}_2 = 0.$

Where $\beta_{1n} = a_{1n}/a_{11}$, n=2,3; $\beta_{m1} = a_{m1}/a_{mm}$, m=2,3

This state would exist only when $k_1 > \beta_{13}k_3$, $k_3 > \beta_{31}k_1$ and $1 - \beta_{13}\beta_{31} > 0$

E_8: $\overline{N}_1 = \frac{k_1 - (\beta_{12}k_2 + \beta_{13}k_3)}{1-(\beta_{21}\beta_{12}+\beta_{31}\beta_{13})}, \overline{N}_2 = \frac{k_2 + \beta_{21}\beta_{13}k_3 - (\beta_{31}\beta_{13}k_2 + \beta_{21}k_1)}{1-(\beta_{21}\beta_{12}+\beta_{31}\beta_{13})}$,

$$\overline{N}_3 = \frac{k_3+\beta_{31}\beta_{12}k_2-(\beta_{21}\beta_{12}k_3+\beta_{31}k_1)}{1-(\beta_{21}\beta_{12}+\beta_{31}\beta_{13})}$$

4. Methodology:

***Stability Analysis*:**

Let $N=(N_1,N_2,N_3)=\overline{N}+U$ (5)

here $U=(u_1,u_2,u_3)^T$ represents a minor deviation from the equilibrium condition $\overline{N}=(\overline{N}_1,\overline{N}_2,\overline{N}_3)$.

The perturbed state equations are obtained by quasi-linearizing the fundamental equations (1), (2), and (3).

$$\frac{dU}{dt} = AU \tag{6}$$

Where

$$A(E)=\begin{bmatrix} a_1-2a_{11}\overline{N}_1-a_{12}\overline{N}_2-a_{13}\overline{N}_3 & -a_{12}\overline{N}_1 & -a_{13}\overline{N}_1 \\ -a_{21}\overline{N}_2 & a_2-2a_{22}\overline{N}_2-a_{21}\overline{N}_1 & 0 \\ -a_{31}\overline{N}_3 & 0 & a_3-2a_{33}\overline{N}_3-a_{31}\overline{N}_1 \end{bmatrix} \tag{7}$$

The system's characteristic equation is $|A-\lambda I|=0$ (8)

If all of the roots of equation (8) are less than zero, if the roots are real, or have negative real parts, if the roots are complex, then the equilibrium state is stable.

4.1. $E_1=(\overline{N}_1,\overline{N}_2,\overline{N}_3)=(0,0,0)$**:**

By taking $\overline{N}_1$, $\overline{N}_2$ & $\overline{N}_3$ values in (7) i.e., in A(E), we get

$$A(E_1)=\begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix} \tag{9}$$

The latent equation is $(\lambda-a_1)(\lambda-a_2)(\lambda-a_3)=0$ (10)

The eigenvalues are a_1, a_2, a_3. Here all the characteristic roots are greater than zero, hence E_1 is **unstable.**

$u_1=u_{10}\,e^{a_1 t}$, $u_2=u_{20}\,e^{a_2 t}$ and $u_3=u_{30}\,e^{a_3 t}$ (11)

are the solutions.

Where u_{20} ,u_{10} ,u_{30} represent the respective starting values of of u_2,u_1,u_3 respectively.

The planes' trajectories are provided by in $u_1 - u_2$, $u_2 - u_3$:

$$\left(\frac{u_1}{u_{10}}\right)^{\frac{1}{a_1}} = \left(\frac{u_2}{u_{20}}\right)^{\frac{1}{a_2}} = \left(\frac{u_3}{u_{30}}\right)^{\frac{1}{a_3}}$$

4.2. E_2 : $\bar{N}_2 = 0$, $\bar{N}_1 = k_1$, $\bar{N}_3 = 0$

By taking $\bar{N}_1$, $\bar{N}_2$ & $\bar{N}_3$ values in (7) i.e., in A(E), we get

$$A(E_2)=\begin{bmatrix} -a_1 & -a_{12}k_1 & -a_{13}k_1 \\ 0 & a_2 - a_{21}k_1 & 0 \\ 0 & 0 & a_3 - a_{31}k_1 \end{bmatrix} \tag{12}$$

The latent roots are $-a_1, a_2 - a_{21}k_1, a_3 - a_{31}k_1$.

If $a_2 < a_{21}k_1$ and $a_3 < a_{31} k_1$, all the characteristic roots are less than zero. Hence E_2 is consistent, otherwise **unstable** , and the solution curves are:

$$u_1 = u_{10}e^{-a_1 t} - \frac{(a_1 a_{12} u_{20})\, e^{(a_2 - k_1 a_{21})t}}{a_1(a_{11}-a_{21})+a_2\, a_{11}} - \frac{(a_1 a_{13} u_{30})\, e^{(a_3 - k_1 a_{31})t}}{a_1(a_{11}-a_{31})+a_3\, a_{11}}\ ;\ u_2 = u_{20}e^{(a_2 - a_{21}k_1)t} \quad \text{and}$$

$$u_3 = u_{30}e^{(a_3 - a_{31}k_1)t} \tag{13}$$

And the curves in phase planes,

$$u_1 = \frac{u_2}{u_{20}}\left\{u_{10}e^{(-a_2 + a_{21}k_1 - a_1)t} - \frac{(a_1 a_{12} u_{20})}{a_1(a_{11}-a_{21})+a_2\, a_{11}} - \frac{(a_1 a_{13} u_{30})e^{(-a_2 + (a_{21} - a_{31})k_1 + a_3)t}}{a_1(a_{11}-a_{31})+a_3\, a_{11}}\right\} \text{ and}$$

$$\left(\frac{u_2}{u_{20}}\right)^{\frac{1}{(a_2 - a_{21}k_1)}} = \left(\frac{u_3}{u_{30}}\right)^{\frac{1}{(a_3 - a_{31}k_1)}}$$

4.3. E_3 : $\bar{N}_1 = 0$, $\bar{N}_3 = 0$, $\bar{N}_2 = k_2$,

In this point,

The roots are $a_1 - a_{12}k_2$, $-a_2$, a_3, here one characteristic root a_3 is non negative so E_3 is **instable** , and the solution set of the equations (1),(2) ,(3):

$$u_1 = u_{10}e^{(a_1 - a_{12}k_2)t},\ u_2 = u_{20}e^{-a_2 t} - \frac{(a_2 a_{21} u_{10})\, e^{(a_1 - k_2 a_{12})t}}{a_2(a_{22}-a_{12})+a_1\, a_{22}}\ \&\ u_3 = u_{30}\, e^{a_3 t} \tag{14}$$

Paths taken by disturbances,

$$u_2 = \frac{u_1}{u_{10}}\left\{u_{20}e^{(a_{12}k_2 - a_2 - a_1)t} - \frac{(a_2 a_{21} u_{10})}{a_2(a_{22}-a_{12})+a_1\, a_{22}}\right\} \text{ and}$$

$$u_2 = \frac{u_3}{u_{30}}\left\{u_{20}e^{-(a_2 + a_3)t} - \frac{(a_2 a_{21} u_{10})\, e^{(a_1 - a_3 - k_2 a_{12})t}}{a_2(a_{22}-a_{12})+a_1\, a_{22}}\right\}$$

4.4. $E_4 : \bar{N}_1 = 0, \bar{N}_2 = 0, \bar{N}_3 = k_3$

Where $a_1 - a_{13}k_3, -a_3, a_2$ are the characteristic roots. The state is unstable, because one of the three characteristic roots are positive. The solutions to the equation are produced,

$$u_1 = u_{10}e^{(a_1 - a_{13}k_3)\,t}, u_2 = u_{20}e^{a_2 t} \ \&\ u_3 = u_{30}e^{-a_3 t} - \frac{(a_3 a_{31} u_{10})\, e^{(a_1 - k_3 a_{13})t}}{a_3(a_{33} - a_{13}) + a_1 a_{33}} \quad (15)$$

The paths in the planes of $u_1 - u_2$ and $u_2 - u_3$ are

$$u_1 = \frac{u_2}{u_{20}}\left\{u_{10}e^{-(a_{13}k_3 + a_2 - a_1)t}\right\} \text{ and } u_3 = \frac{u_2}{u_{20}}\left\{u_{30}e^{-(a_3 + a_2)t} - \frac{(a_3 a_{31} u_{10})\, e^{(a_1 - a_2 - k_3 a_{13})t}}{a_3(a_{33} - a_{13}) + a_1 a_{33}}\right\}$$

4.5. $E_5 : \bar{N}_1 = 0, \bar{N}_2 = k_2, \bar{N}_3 = k_3$

The latent values are $a_1 - (a_{12}k_2 + a_{13}k_3), -a_2, -a_3$, here two eigen values are negative, stability depends on third eigen value.

If $a_1 > a_{12}k_2 + a_{13}k_3$,the eigen value is positive then E_5 is **unstable.**

If $a_1 < a_{12}k_2 + a_{13}k_3$,the eigen value is negative then E_5 is **stable.**

And the set of solutions of (1),(2) ,(3) are:

$$u_1 = u_{10}e^{S_1 t}, \quad u_2 = u_{20}e^{-a_2 t} + \frac{(k_2 a_{21} u_{10})\,(e^{-a_2 t} - e^{S_1 t})}{a_1 + a_2 - a_{12}k_2 - a_{13}k_3},$$

$$u_3 = u_{30}e^{-a_3 t} + \frac{(k_3 a_{31} u_{10})\,(e^{-a_3 t} - e^{S_1 t})}{a_1 + a_2 - a_{12}k_2 - a_{13}k_3} \quad (16)$$

where $S_1 = a_1 - a_{12}\bar{N}_2 - a_{13}\bar{N}_3$

$$u_2 = \frac{u_1}{u_{10}}\left\{u_{20}e^{-(a_2 + S_1)t} + \frac{(k_2 a_{21} u_{10})\,(e^{-(a_2 + S_1)t} - 1)}{a_1 + a_2 - a_{12}k_2 - a_{13}k_3}\right\}$$

$$u_3 = u_2\left[\frac{u_{30}e^{-a_3 t}(a_1 + a_2 - a_{12}k_2 - a_{13}k_3) + (k_3 a_{31} u_{10})\,(e^{-a_3 t} - e^{S_1 t})}{u_{20}e^{-a_2 t}(a_1 + a_2 - a_{12}k_2 - a_{13}k_3) + (k_2 a_{21} u_{10})\,(e^{-a_2 t} - e^{S_1 t})}\right] \text{ are the paths.}$$

4.6. Equilibrium state : $E_6 : \bar{N}_1 = \frac{k_1 - \beta_{12}k_2}{1 - \beta_{21}\beta_{12}}, \bar{N}_2 = \frac{k_2 - \beta_{21}k_1}{1 - \beta_{21}\beta_{12}}, \bar{N}_3 = 0$

After applying $|A(E_6) - \lambda I| = 0$,

One of the eigen value $a_3 - a_{31}\overline{N_1}$ is negative if $a_3 < a_{31}\overline{N_1}$ and other two are given by the quadratic equation

$$\lambda^2 + (a_{11}\overline{N_1} + a_{22}\overline{N_2})\lambda + (a_{11}a_{22} - a_{12}a_{21})(\overline{N_1})(\overline{N_2}) = 0 \quad (17)$$

Whose sum of roots $-(a_{11}\overline{N_1} + a_{22}\overline{N_2})$ is negative only when $(a_1 a_{22} + a_2 a_{11}) > (a_2 a_{12} + a_1 a_{21})$ & $a_{11}a_{22} > a_{12}a_{21}$, product of the roots $(a_{11}a_{22} - a_{12}a_{21})(\overline{N_1})(\overline{N_2})$ is positive only when $a_1a_2 > (a^2{}_1a_{21}a_{22} + a^2{}_2a_{12}a_{11})$ & $a_{11}a_{22} > a_{12}a_{21}$ .Therefore the roots of above equation is real and negative or Complex conjugates having real parts .**Thus the state is asymptotically stable if** $a_3 < a_{31}\overline{N_1}$, $(a_1a_{22} + a_2a_{11}) > (a_2a_{12} + a_1a_{21})$, $a_1a_2 > (a^2{}_1a_{21}a_{22} + a^2{}_2a_{12}a_{11})$ & $a_{11}a_{22} > a_{12}a_{21}$.

And the set of solutions of (1) ,(2) ,(3) are:

$$u_1 = u_{10}\left(A_0e^{\lambda_1 t} + B_0e^{\lambda_2 t} + C_0e^{\lambda_3 t}\right), u_2 = u_{20}\left(A_1e^{\lambda_1 t} + B_1e^{\lambda_2 t} + C_1e^{\lambda_3 t}\right) \& u_3 = u_{30}e^{\lambda_1 t}$$

Where $\lambda_1 = a_3 - a_{31}\overline{N_1}$, λ_2 and λ_3 are eigen roots of (17),

$$A_0 = \frac{\lambda_1{}^2 - P_1\lambda_1 + Q_1}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}, B_0 = \frac{\lambda_2{}^2 - P_1\lambda_2 + Q_1}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)}, C_0 = \frac{\lambda_3{}^2 - P_1\lambda_3 + Q_1}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}$$

$$A_1 = \frac{\lambda_1{}^2 + P_2\lambda_1 - Q_2}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}, B_1 = \frac{\lambda_2{}^2 + P_2\lambda_2 - Q_2}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)}, C_1 = \frac{\lambda_3{}^2 + P_2\lambda_3 - Q_2}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}$$

$$P_1 = \lambda_1 - a_{22}\overline{N_2} + \frac{(u_{30}a_{13} + u_{20}a_{12})}{u_{10}}\overline{N_1}, P_2 = a_{11}\overline{N_1} - \lambda_1 + \frac{(u_{10}a_{12})}{u_{20}}\overline{N_2},$$

$$Q_1 = -a_{22}\overline{N_2}\lambda_1 + \frac{(u_{20}a_{12}\lambda_1 - u_{30}a_{13}a_{22}\overline{N_2})\overline{N_1}}{u_{10}}, Q_2 = a_{11}\overline{N_1}\lambda_1 - \frac{(u_{10}a_{12}\lambda_1 + u_{30}a_{12}a_{13}\overline{N_1})\overline{N_2}}{u_{20}}.$$

4.7. E_7 : $\overline{N_1} = \frac{a_1\,a_{33} - a_3\,a_{13}}{a_{11}a_{33} - a_{13}a_{31}}$, $\overline{N_2} = 0$, $\overline{N_3} = \frac{a_3\,a_{11} - a_1\,a_{31}}{a_{11}a_{33} - a_{13}a_{31}}$

After applying $|A(E_7) - \lambda I| = 0$,

One of the eigen value is $a_2 - a_{21}\overline{N_1}$ is negative if $a_2 < a_{21}\overline{N_1}$ and other two are given by the quadratic equation

$$\lambda^2 + (a_{11}\overline{N_1} + a_{33}\overline{N_3})\lambda + (a_{11}a_{33} - a_{13}a_{31})(\overline{N_1})(\overline{N_3}) = 0 \qquad (18)$$

Whose sum of roots $-(a_{11}\overline{N_1} + a_{33}\overline{N_3})$ is negative only when $(a_1a_{33} + a_3a_{11}) > (a_3a_{13} + a_1a_{31})$ & $a_{11}a_{33} > a_{13}a_{31}$, product of the roots $(a_{11}a_{33} - a_{13}a_{31})(\overline{N_1})(\overline{N_3})$ is positive only when $a_1a_3 > (a^2{}_1a_{31}a_{33} + a^2{}_3a_{13}a_{11})$ & $a_{11}a_{33} > a_{13}a_{31}$ .Therefore the roots of above equation is real and negative or Complex conjugates having real parts .**Thus the state is asymptotically stable if** $a_2 < a_{21}\overline{N_1}$, $(a_1a_{33} + a_3a_{11}) > (a_3a_{13} + a_1a_{31})$, $a_1a_3 > (a^2{}_1a_{31}a_{33} + a^2{}_3a_{13}a_{11})$ and $a_{11}a_{33} > a_{13}a_{31}$.

And the set of solutions of (1) ,(2) ,(3) are:

$$u_1 = u_{10}\left(A_2e^{\lambda_1 t} + B_2e^{\lambda_2 t} + C_2e^{\lambda_3 t}\right), u_2 = u_{20}e^{\lambda_1 t} \& u_3 = u_{30}\left(A_3e^{\lambda_1 t} + B_3e^{\lambda_2 t} + C_3e^{\lambda_3 t}\right)$$

Where $\lambda_1 = a_2 - a_{21}\overline{N_1}$, λ_2 and λ_3 are eigen roots of eq (18)

$$A_2 = \frac{\lambda_1{}^2 + P_3\lambda_1 + Q_3}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}, B_2 = \frac{\lambda_2{}^2 + P_3\lambda_2 + Q_3}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)}, C_2 = \frac{\lambda_3{}^2 + P_3\lambda_3 + Q_3}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}$$

$A_3 = \frac{\lambda_1{}^2+P_4\lambda_1-Q_4}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}$, $B_3 = \frac{\lambda_2{}^2+P_4\lambda_2-Q_4}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)}$, $C_3 = \frac{\lambda_3{}^2+P_4\lambda_3-Q_4}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}$

$P_3 = a_{33}\overline{N_3} - \lambda_1 - \frac{(u_{20}a_{12}+u_{30}a_{13})}{u_{10}}\overline{N_1}$, $P_4 = a_{11}\overline{N_1} - \lambda_1 - \frac{(u_{10}a_{31})}{u_{30}}\overline{N_3}$,

$Q_3 = -a_{33}\overline{N_3}\lambda_1 + \frac{(u_{30}a_{13}\lambda_1-u_{20}a_{12}a_{33}\overline{N_3})\overline{N_1}}{u_{10}}$, $Q_4 = a_{11}\overline{N_1}\lambda_1 + \frac{(u_{10}a_{31}\lambda_1+u_{30}a_{12}a_{31}\overline{N_1})\overline{N_3}}{u_{30}}$.

4.8. Stability of the equilibrium state E_8:

After applying $|A(E_8)-\lambda I|=0$ the characteristic equation can be

$\lambda^3 + b_1\lambda^2 + b_2\lambda + b_3 = 0$

Where $b_1 = a_{11}\overline{N}_1 + a_{22}\overline{N}_2 + a_{33}\overline{N}_3$,

$b_2 = a_{22}a_{33}\overline{N}_2\overline{N}_3 + (a_{11}a_{22} - a_{12}a_{21})\overline{N}_1\overline{N}_2 + (a_{11}a_{33} - a_{13}a_{31})\overline{N}_1\overline{N}_3$,

$b_3 = (a_{11}a_{22}a_{33} - a_{12}a_{21}a_{33} - a_{13}a_{31}a_{22})\overline{N}_1\overline{N}_2\overline{N}_3$

According to Routh-Hurwitz's criteria, the necessary and sufficient conditions for local stability of co-existent points are $b_1 > 0, b_3 > 0$ & $b_3(b_1b_2 - b_3) > 0$

It is evident that $b_1 > 0$ and $a_{11}a_{22}a_{33}\overline{N}_1\overline{N}_2\overline{N}_3 > (a_{12}a_{21}a_{33} + a_{13}a_{31}a_{22})\overline{N}_1\overline{N}_2\overline{N}_3$ .Thus the stability of co-existent state is determined by the sign of $b_1b_2 - b_3$

By the calculations we obtain $b_1b_2 - b_3 = (a^2{}_{11}a_{22} - a_{11}a_{12}a_{21})\overline{N}_1{}^2\overline{N}_2 + (a^2{}_{11}a_{33} - a_{11}a_{13}a_{31})\overline{N}_1{}^2\overline{N}_3 + a^2{}_{22}a_{33}\overline{N}_2{}^2\overline{N}_3 + (a^2{}_{22}a_{11} - a_{12}a_{21}a_{22})\overline{N}_2{}^2\overline{N}_1 + a^2{}_{33}a_{22}\overline{N}_3{}^2\overline{N}_2 + (a^2{}_{33}a_{11} - a_{13}a_{31}a_{33})\overline{N}_3{}^2\overline{N}_1 + 2a_{11}a_{22}a_{33}\overline{N}_1\overline{N}_2\overline{N}_3 > 0$

Hence the co-existent state is **locally asymptotically stable**.

The solution of the perturbation equation is

$u_1 = u_{10}\left(A_4e^{\lambda_1 t} + B_4e^{\lambda_2 t} + C_4e^{\lambda_3 t}\right)$, $u_2 = u_{20}\left(A_5e^{\lambda_1 t} + B_5e^{\lambda_2 t} + C_5e^{\lambda_3 t}\right)$ and

$$u_3 = u_{30}\left(A_6e^{\lambda_1 t} + B_6e^{\lambda_2 t} + C_6e^{\lambda_3 t}\right) \quad (19)$$

Where

$A_4 = \frac{\lambda_1{}^2+P_5\lambda_1+Q_5}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}$, $B_4 = \frac{\lambda_2{}^2+P_5\lambda_2+Q_5}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)}$, $C_4 = \frac{\lambda_3{}^2+P_5\lambda_3+Q_5}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}$

$A_5 = \frac{\lambda_1{}^2+P_6\lambda_1+Q_6}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}$, $B_5 = \frac{\lambda_2{}^2+P_6\lambda_2+Q_6}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)}$, $C_5 = \frac{\lambda_3{}^2+P_6\lambda_3+Q_6}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}$

$A_6 = \frac{\lambda_1{}^2+P_7\lambda_1+Q_7}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}$, $B_6 = \frac{\lambda_2{}^2+P_7\lambda_2+Q_7}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)}$, $C_6 = \frac{\lambda_3{}^2+P_7\lambda_3+Q_7}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}$

$P_5 = (a_{22}\overline{N_2} + a_{33}\overline{N_3}) - \frac{(u_{20}a_{12}+u_{30}a_{13})}{u_{10}}\overline{N_1}$;$P_6 = (a_{33}\overline{N_3} + a_{11}\overline{N_1}) - \frac{u_{10}a_{21}}{u_{20}}\overline{N_2}$;

$P_7 = (a_{11}\overline{N_1} + a_{22}\overline{N_2}) - \frac{u_{10}a_{31}}{u_{30}}\overline{N_3}$; $Q_5 = a_{22}a_{33}\bar{N}_2\bar{N}_3 - \frac{(u_{20}a_{12}a_{33}\bar{N}_3+u_{30}a_{13}a_{22}\bar{N}_2)}{u_{10}}\bar{N}_1$;

$Q_6 = (a_{11}a_{33} - a_{31}a_{13})\bar{N}_1\bar{N}_3 + \frac{(u_{30}a_{13}a_{21}\bar{N}_1 - u_{10}a_{33}a_{21}\bar{N}_3)}{u_{20}}\bar{N}_2$;

$Q_7 = (a_{11}a_{22} - a_{21}a_{12})\bar{N}_1\bar{N}_2 + \frac{(u_{20}a_{12}a_{31}\bar{N}_1 - u_{10}a_{31}a_{22}\bar{N}_2)}{u_{30}}\bar{N}_3$.

5. Lyapunov's Mapping for Global Stability:

We have covered the stability at local stage, each of the eight equilibrium states in section 4. whereas the other states are unstable and the first five are stable. Using the appropriate Lyapunov's functions, now we investigate the global stability of dynamical systems (1), (2), and (3) at these five stages.

Theorem 1: Asymptotically, the equilibrium state $E_2(\bar{N}_1, 0,0)$ is universally stable.

Proof: We choose the Liapunov's function as

$$W(N_1, N_2, N_3) = \left\{N_1 - \bar{N}_1 - \bar{N}_1 \ln \left\{\frac{N_1}{\bar{N}_1}\right\}\right\} \tag{20}$$

$$\frac{dW}{dt} = \left[1 - \frac{\bar{N}_1}{N_1}\right]\frac{dN_1}{dt} = -a_{11}(N_1 - \bar{N}_1)^2 < 0$$

Thus, the system is stable globally asymptotically.

Theorem 2: Asymptotically, the point $E_2(\bar{N}_1, 0,0)$ is globally stable.

Proof: Take the Lyapunov's function as follows

$$W(N_2, N_3) = \left\{N_2 - \bar{N}_2 - \bar{N}_2 \ln \left\{\frac{N_2}{\bar{N}_2}\right\}\right\} + \left\{N_3 - \bar{N}_3 - \bar{N}_3 \ln \left\{\frac{N_3}{\bar{N}_3}\right\}\right\} \tag{21}$$

$$\frac{dW}{dt} = \left[1 - \frac{\bar{N}_2}{N_2}\right]\frac{dN_2}{dt} + \left[1 - \frac{\bar{N}_3}{N_3}\right]\frac{dN_3}{dt}$$

$$\frac{dW}{dt} = (N_2 - \bar{N}_2)(a_2 - a_{22}N_2) + (N_3 - \bar{N}_3)(a_3 - a_{33}N_3)$$

$$\frac{dW}{dt} = -\{a_{22}(N_2 - \bar{N}_2)^2 + a_{33}(N_3 - \bar{N}_3)^2\}$$

$$\frac{dW}{dt} < 0$$

Hence, the System is Globally Asymptotically Stable.

Theorem 3: The balance state $E_6(\bar{N}_1, \bar{N}_2, 0)$ is globally asymptotically balanced.

Proof : Define the following Lyapunov's mapping as

$$W(N_1, N_2) = \left\{N_1 - \bar{N}_1 - \bar{N}_1 \ln \left\{\frac{N_1}{\bar{N}_1}\right\}\right\} + \left\{N_2 - \bar{N}_2 - \bar{N}_2 \ln \left\{\frac{N_2}{\bar{N}_2}\right\}\right\} \quad (23)$$

$$\frac{dW}{dt} = \left[1 - \frac{\bar{N}_1}{N_1}\right]\frac{dN_1}{dt} + \left[1 - \frac{\bar{N}_2}{N_2}\right]\frac{dN_2}{dt}$$

$$\frac{dW}{dt} = \{(a_1 - a_{11}N_1 - a_{12}N_2)(N_1 - \bar{N}_1) + (a_2 - a_{22}N_2 - a_{21}N_1)(N_2 - \bar{N}_2)\}$$

$$\frac{dW}{dt} = -\{a_{11}(N_1 - \bar{N}_1)^2 + a_{22}(N_2 - \bar{N}_2)^2 + (a_{12} + a_{21})(N_1 - \bar{N}_1)(N_2 - \bar{N}_2)\}$$

$\frac{dW}{dt} < 0$. When $a_{12} = -a_{21}$

Hence the System is Globally Asymptotically Stable.

Theorem 4: The balanced point E_6 is universally asymptotically stable.

Proof: The Liapunov's function is defined by

$$W(N_1, N_3) = \left\{N_1 - \bar{N}_1 - \bar{N}_1 \ln \left\{\frac{N_1}{\bar{N}_1}\right\}\right\} + \left\{N_3 - \bar{N}_3 - \bar{N}_3 \ln \left\{\frac{N_3}{\bar{N}_3}\right\}\right\} \quad (24)$$

$$\frac{dW}{dt} = \left[1 - \frac{\bar{N}_1}{N_1}\right]\frac{dN_1}{dt} + \left[1 - \frac{\bar{N}_3}{N_3}\right]\frac{dN_3}{dt}$$

$$\frac{dW}{dt} = \{(N_1 - \bar{N}_1)(a_1 - a_{11}N_1 - a_{13}N_3) + (N_3 - \bar{N}_3)(a_3 - a_{33}N_3 - a_{31}N_1)\}$$

$$\frac{dW}{dt} = -\{a_{11}(N_1 - \bar{N}_1)^2 + a_{33}(N_3 - \bar{N}_3)^2 + (a_{13} + a_{31})(N_1 - \bar{N}_1)(N_3 - \bar{N}_3)\}$$

$\frac{dW}{dt} < 0$. When $a_{13} = -a_{31}$

Hence, the System is Globally Asymptotically Stable.

Theorem 5: The Normal Steady State is balanced universally

Proof : Revolve the mapping,

$$W(N_1, N_2, N_3) = N_1 - \bar{N}_1 - \bar{N}_1 \ln\left\{\frac{N_1}{\bar{N}_1}\right\} + N_2 - \bar{N}_2 - \bar{N}_2 \ln\left\{\frac{N_2}{\bar{N}_2}\right\} + N_3 - \bar{N}_3 - \bar{N}_3 \ln\left\{\frac{N_3}{\bar{N}_3}\right\} \quad (25)$$

$$\frac{dW}{dt} = \left[1 - \frac{\bar{N}_1}{N_1}\right]\frac{dN_1}{dt} + \left[1 - \frac{\bar{N}_2}{N_2}\right]\frac{dN_2}{dt} + \left[1 - \frac{\bar{N}_3}{N_3}\right]\frac{dN_3}{dt}$$

$$\frac{dW}{dt} = \{(N_1 - \bar{N}_1)(a_1 - a_{11}N_1 - a_{12}N_2 - a_{13}N_3) + (N_2 - \bar{N}_2)(a_2 - a_{22}N_2 - a_{21}N_1) + (N_3 - \bar{N}_3)(a_3 - a_{33}N_3 - a_{31}N_1)\}$$

$$\frac{dW}{dt} = -\{a_{11}(N_1 - \bar{N}_1)^2 + a_{22}(N_2 - \bar{N}_2)^2 + a_{33}(N_3 - \bar{N}_3)^2 + (N_1 - \bar{N}_1)(N_2 - \bar{N}_2)(a_{12} + a_{21}) + (N_1 - \bar{N}_1)(N_3 - \bar{N}_3)(a_{13} + a_{31})\}$$

$\frac{dW}{dt} < 0$. When $a_{12} = -a_{21}$ and $a_{13} = -a_{31}$

Hence, E_8 is universally asymptotically balanced.

6. A numerical Approach of the Growth Rate Equations

The numerical solutions of the growth rate equations (1), (2) and (3) computed employing the fourth order Runge-Kutta method for specific values of the various parameters that characterize the model and the initial conditions. The results are illustrated in Figures 1 to 8.

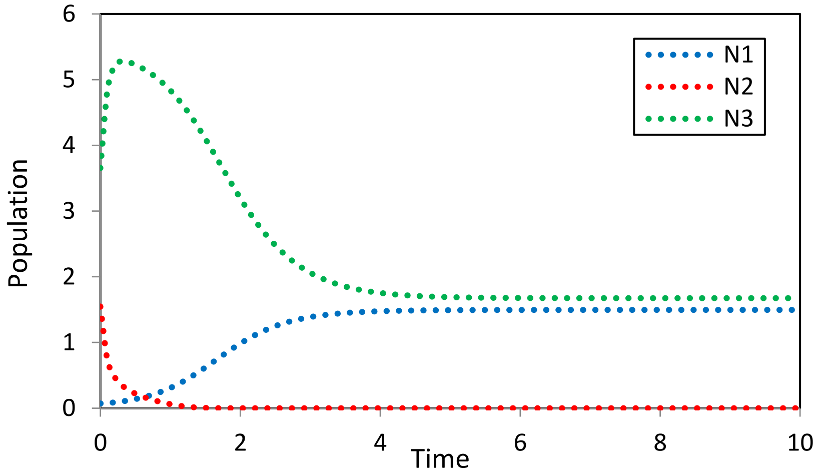

Figure 1: a_1=3.186, a_2=1.404, a_3=15.192, a_{11}=1.908, a_{12}=1.332, a_{13}=0.198, a_{22}=7.524, a_{21}=13.986, a_{33}=2.736, a_{31}=7.092, N_1=0.072, N_2=1.548, N_3=3.654.

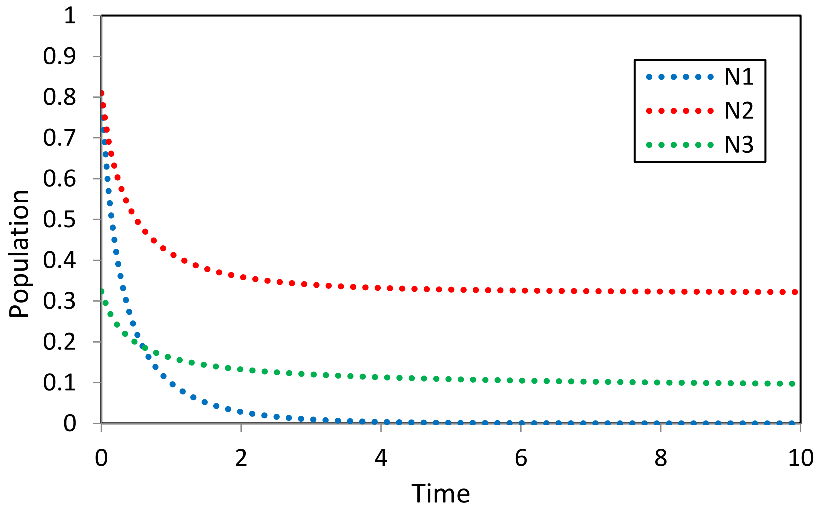

Figure 2: a_1=0.54, a_2=0.306, a_3=0.18, a_{11}=0.72, a_{12}=4.374, a_{13}=0.54, a_{22}=0.954, a_{21}=1.620, a_{33}=1.98, a_{31}=1.620, N_1=0.81, N_2=0.81, N_3=0.324.

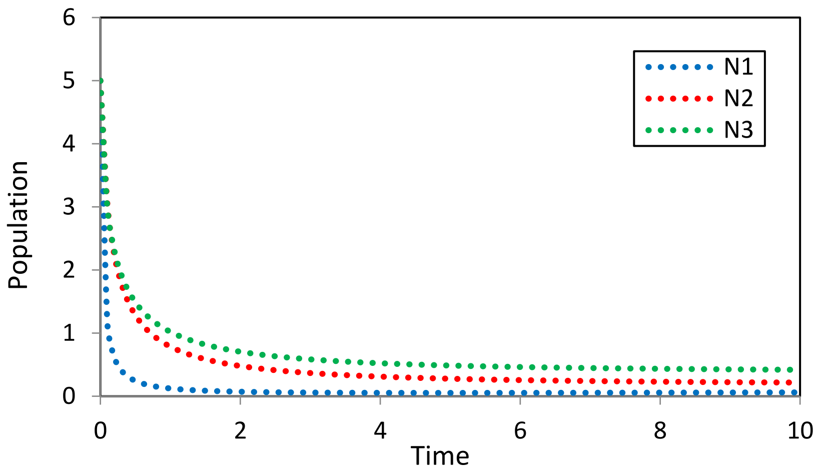

Figure 3: a_1=0.576, a_2=0.252, a_3=0.378, a_{11}=4.788, a_{12}=1.08, a_{13}=0.05, a_{22}=1.206, a_{21}=0.252, a_{33}=0.774, a_{31}=1.152, N_1=5.0, N_2=5.0, N_3=5.0.

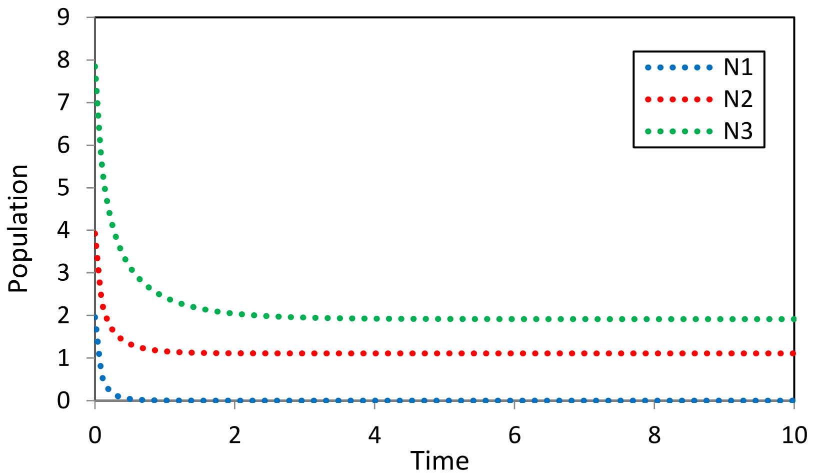

Figure 4: a_1=0.378, a_2=2.556, a_3=1.206, a_{11}=0.828, a_{12}=0.63, a_{13}=1.476, a_{22}=2.304, a_{21}=0.954, a_{33}=0.63, a_{31}=0.576, N_1=1.962, N_2=3.924, N_3=7.848.

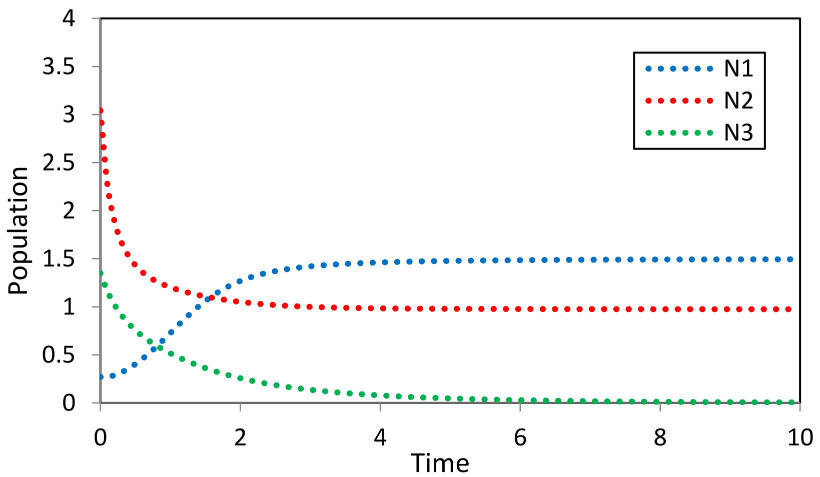

Figure 5: a_1=4.014, a_2=2.466, a_3=0.504, a_{11}=1.98, a_{12}=1.08, a_{13}=0.54, a_{22}=2.034, a_{21}=0.324, a_{33}=1.476, a_{31}=0.63, N_1=0.27, N_2=3.042, N_3=1.35.

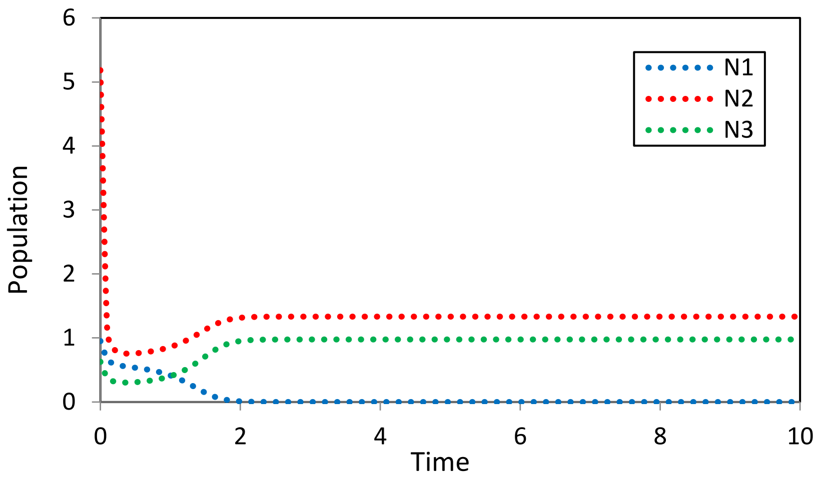

Figure 6: a_1=6.606, a_2=14.814, a_3=12.186, a_{11}=4.986, a_{12}=0.054, a_{13}=13.446, a_{22}=11.106, a_{21}=11.7, a_{33}=12.474, a_{31}=15.12, N_1=0.954, N_2=5.184, N_3=0.63.

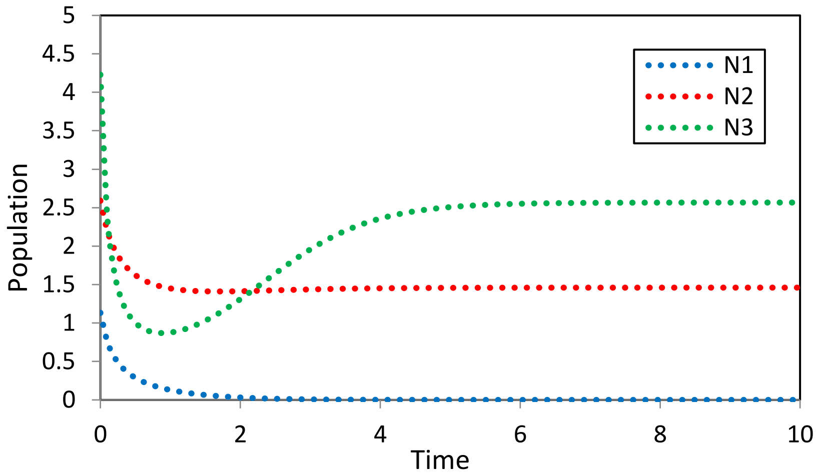

Figure 7: a_1=0.774, a_2=1.314, a_3=1.386, a_{11}=0.954, a_{12}=1.26, a_{13}=0.306, a_{22}=0.9, a_{21}=0.9, a_{33}=0.54, a_{31}=5.994, N_1=1.134, N_2=2.592, N_3=4.23.

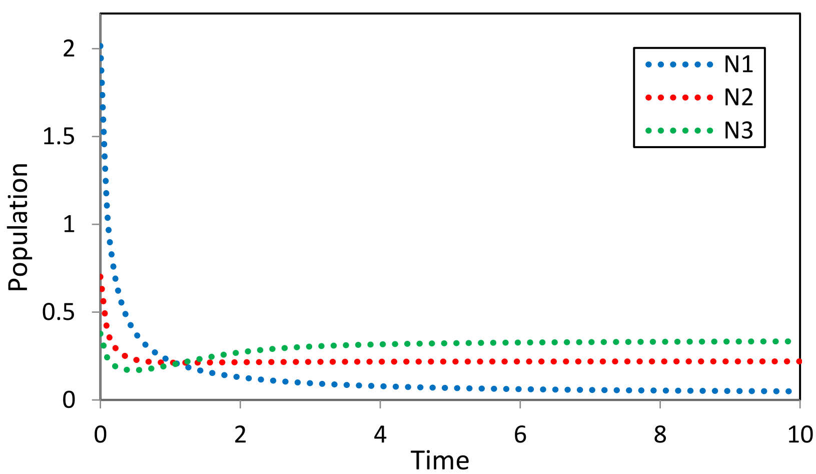

Figure 8: a_1=0.36, a_2=3.546, a_3=2.7, a_{11}=4.5, a_{12}=0.594, a_{13}=0.126, a_{22}=15.966, a_{21}=0.774, a_{33}=7.56, a_{31}=3.546, N_1=2.016, N_2=0.702, N_3=0.378.

7. Discussions and Conclusion

Case i): The population of S_1 increased whereas the populations of S_2, S_3 are decreased. And the populations of S_1 and S_3 are almost equal after $t^*=4$. All are illustrated in figure 1.

Case ii): The growth rates of S_1,S_2 and S_3 are in decreasing order. The populations of $S_1$and S_3 coincides at $t^*=0.4$.The population of S_1 almost vanishes after $t^*=3$ which are illustrated in figure 2.

Case iii): The Initial conditions of S_1,S_2 and S_3 are same. Populations are decreased as time passes, which are illustrated in figure 3.

Case iv): The self-inhibition coefficients of $S_1$and S_3 are almost same and initial conditions of S_1,S_2 and S_3 are in increasing order. The population of S_1 almost vanishes after $t^*=0.5$, which are illustrated in figure 4.

Case v): The first species was dominated by second and third initially, but later the first species dominated the remaining two. The growth rates of S_2 and S_3 are in decreasing mode whereas S_1 is in increasing mode. Third species vanishes almost after $t^*=6$ we can observe in figure 5.

Case vi): The self-inhibition coefficients of S_1,S_2 and S_3 are in increasing mode. From $t^*=2$ both the species population S_2 and S_3 are almost same and S_1 vanishes. We can see in figure 6.

Case vii): The Initial conditions and growth rates of S_1,S_2 and S_3 are in increasing mode. First species vanishes at $t^*=2.2$ and S_2 and S_3 coincides at $t^*=2.2$ then S_3 dominates S_2 . we can observe in figure 7.

Case viii): The Initial conditions of S_1,S_2 and S_3 are in decreasing mode. The populations of all species coincide at $t^*=1$ approximately. Third species dominates the remaining two and second dominates the first, first species is dominated by the remaining two. we can observe in figure 8.

This research examines a continuous model of a resource-constrained, three-species competitive ecology. Three first ODEs make up the set of model equations. The model's potential equilibrium states are all identified, and the stability requirements for each are covered. It is found that the equilibrium states E_2, E_5, E_6, E_7, and E_8 are stable in all states. Additionally, using a properly constructed Liapunov's function, there has been established a global stability of the system, and the trajectories of the perturbations across the equilibrium points are depicted.

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