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A Set of Integrals Associated With the Incomplete ℵ-Function, Multi-Index Bessel Maitland Function and Jacobi Polynomial

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ABSTRACT:

The Bessel-Maitland function and the generalised Bessel-Maitland function have recently attracted the attention of numerous authors. Integral transformations and fractional calculus results involving these functions have been demonstrated by a number of researchers. The main key of the paper, we establish some finite and infinite integral formulas containing the product of multi-index Bessel Maitland function, incomplete ℵ-function(IAF) and Jacobi polynomial which we have defined in this article. Further, we establish many interesting corollaries by specializing the parameters of IAF in the form of simpler special functions like incomplete H-function, incomplete Wright hypergeometric function, incomplete Fox-Wright function, Fox-Wright function, generalized Mittag-Leffler function etc. with the help of main results.

Keywords And Phrases: Multi-Index Bessel Maitland Function, Incomplete Aleph Function (IAF), Incomplete Fox-Wright Function, Mellin-Barnes Type Contour.
2010 Mathematics Subject Classification: Primary 26A33, 33B20; Secondary 33C60, 33E20, 44A40.1.

1. Introduction and Preliminaries

Various authors have done work on incomplete Aleph function. Several researchers have been established integral transforms and fractional calculus involving these functions and are evaluated and examine number of integral formulas elaborated with the various kind of special function included incomplete Aleph function. Such type of integral formulas has been played an important role in the applied sciences, solving scientific and engineering problems. In 19th century, Edward Maitland Wright [8] introduced the Bessel-Maitland function which is the extended version of Bessel function. Recently, Singh et al. [11] introduced and defined the generalised Bessel-Maitland function. Saxena and Nishimoto [10] defined and studied the multi-index Mittag-Leffler function which is equivalent to multi-index Bessel-Maitland function.

Very recently, Bansal et. al [1] introduced and investigate the incomplete Aleph function $(\Gamma)\aleph^{m,n}_{p_i,q_i,\rho_i,r}(z)$ and $(\gamma)\aleph^{m,n}_{p_i,q_i,\rho_i,r}(z)$ which are defined by Mellin-Barnes type contour integral representations as follows:

$$(\Gamma)\aleph^{m,n}_{p_i,q_i,\rho_i,r}(z) = {}^{(\Gamma)}(\aleph)^{m,n}_{p_i,q_i,\rho_i,r}\left[z\left|\begin{matrix}(a_1,A_1,y),(a_j,A_j)_{2,n},[\rho_j(a_{ji},A_{ji})]_{n+1,p_1}\\ (b_j,B_j)_{1,m},[\rho_j(b_{ji},B_{ji})]_{m+1,q_1}\end{matrix}\right.\right]$$

$$= \frac{1}{2\pi i}\int_{\mathbb{C}} K\,(\xi,y)(z)^{-\xi}d\xi, \tag{1.1}$$

where $z \neq 0$, and

$$K(\xi,y) = \frac{\Gamma(1-a_1-A_1\xi,y)\prod_{j=1}^{m}\Gamma(b_j+B_j\xi)\prod_{j=2}^{n}\Gamma(1-b_1-B_1\xi,y)}{\sum_{i=1}^{r}\rho_i\left[\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}+B_{ji}\xi)\prod_{j=n+1}^{p_i}\Gamma(a_{ji}-A_{ji}\xi)\right]} \tag{1.2}$$

and

$$(\gamma)\aleph^{m,n}_{p_i,q_i,\rho_i,r}(z) = {}^{(\gamma)}(\aleph)^{m,n}_{p_i,q_i,\rho_i,r}\left[z\left|\begin{matrix}(a_1,A_1,y),(a_j,A_j)_{2,n},[\rho_j(a_{ji},A_{ji})]_{n+1,p_1}\\ (b_j,B_j)_{1,m},[\rho_j(b_{ji},B_{ji})]_{m+1,q_1}\end{matrix}\right.\right]$$

$$= \frac{1}{2\pi i}\int_{\mathbb{C}} L\,(\xi,y)(z)^{-\xi}d\xi, \tag{1.3}$$

where $z \neq 0$, and

$$L(\xi,y) = \frac{\gamma(1-a_1-A_1\xi,y)\prod_{j=1}^{m}\Gamma(b_j+B_j\xi)\prod_{j=2}^{n}\Gamma(1-b_1-B_1\xi,y)}{\sum_{i=1}^{r}\rho_i\left[\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}+B_{ji}\xi)\prod_{j=n+1}^{p_i}\Gamma(a_{ji}-A_{ji}\xi)\right]} \tag{1.4}$$

The incomplete $\aleph$−functions given in (1.1)-(1.4) exist for all $y \geq 0$ under the same contour and the same set of conditions as stated in [1]. A complete details of (1.1) to (1.4) can be found in [1].

We recall the classical definition of incomplete Gamma functions $\gamma(\nu,z)$ and $\Gamma(\nu,z)$ is defined as follows:

$$\gamma(\nu,z) = \int_0^z t^{\nu-1}e^{-t}dt \qquad (\mathcal{R}(\nu) > 0; z \geq 0), \tag{1.5}$$

and

$$\Gamma(\nu,z) = \int_z^\infty t^{\nu-1}e^{-t}dt \qquad (\mathcal{R}(\nu) > 0; z \geq 0\ when\ z = 0) \tag{1.6}$$

The incomplete gamma functions $\Gamma(\nu,z)$ and $\gamma(\nu,z)$ are satisfying decomposition formula:

$$\gamma(\nu,z) + \Gamma(\nu,z) = \Gamma(\nu), \qquad (\mathcal{R} <(\nu) > 0) \tag{1.7}$$

Some important particular cases of incomplete ℵ- functions are listed below:

(i) On taking The case y = 0, then (1.2) reduce to the ℵ- functions which was given in [2,3] as follows:

$$ {}^{(\Gamma)}(\aleph)^{m,n}_{p_i,q_i,\rho_i,r}\left[z\,\middle|\,\begin{matrix}(a_1,A_1,0),(a_j,A_j)_{2,n},[\rho_j(a_{ji},A_{ji})]_{n+1,p_1}\\ (b_j,B_j)_{1,m},[\rho_j(b_{ji},B_{ji})]_{m+1,q_1}\end{matrix}\right] $$

$$ = \aleph^{m,n}_{p_i,q_i,\rho_i,r}\left[z\,\middle|\,\begin{matrix}(a_1,A_1)_{1,n},,[\rho_j(a_{ji},A_{ji})]_{n+1,p_1}\\ (b_j,B_j)_{1,m},[\rho_j(b_{ji},B_{ji})]_{m+1,q_1}\end{matrix}\right] \qquad (1.8) $$

(ii) If we take $\rho_i = 1$ in both the (1.2) and (1.4) the function reduce to the incomplete I-function [2].

$$ {}^{(\Gamma)}(\aleph)^{m,n}_{p_i,q_i,\rho_i,r}\left[z\,\middle|\,\begin{matrix}(a_1,A_1,0),(a_j,A_j)_{2,n},[(a_{ji},A_{ji})]_{n+1,p_1}\\ (b_j,B_j)_{1,m},[(b_{ji},B_{ji})]_{m+1,q_1}\end{matrix}\right] $$

$$ = {}^{(\Gamma)}(I)^{m,n}_{p_i,q_i,\rho_i,r}\left[z\,\middle|\,\begin{matrix}(a_1,A_1,y)_{1,n},(a_j,A_j)_{2,n},[(a_{ji},A_{ji})]_{n+1,p_1}\\ (b_j,B_j)_{1,m},[(b_{ji},B_{ji})]_{m+1,q_1}\end{matrix}\right] \qquad (1.9) $$

and

$$ {}^{(\gamma)}(\aleph)^{m,n}_{p_i,q_i,\rho_i,r}\left[z\,\middle|\,\begin{matrix}(a_1,A_1,0),(a_j,A_j)_{2,n},[(a_{ji},A_{ji})]_{n+1,p_1}\\ (b_j,B_j)_{1,m},[(b_{ji},B_{ji})]_{m+1,q_1}\end{matrix}\right] $$

$$ = {}^{(\gamma)}(I)^{m,n}_{p_i,q_i,\rho_i,r}\left[z\,\middle|\,\begin{matrix}(a_1,A_1,y)_{1,n},(a_j,A_j)_{2,n},[(a_{ji},A_{ji})]_{n+1,p_1}\\ (b_j,B_j)_{1,m},[(b_{ji},B_{ji})]_{m+1,q_1}\end{matrix}\right] \qquad (1.10) $$

(iii) For $\rho_i = 1\ and\ y = 0$ in (1.2), then the moderate form in terms of I-function examine by Saxena [9]:

$$ {}^{(\Gamma)}(\aleph)^{m,n}_{p_i,q_i,1,r}\left[z\,\middle|\,\begin{matrix}(a_1,A_1,0),(a_j,A_j)_{2,n},[(a_{ji},A_{ji})]_{n+1,p_1}\\ (b_j,B_j)_{1,m},[(b_{ji},B_{ji})]_{m+1,q_1}\end{matrix}\right] $$

$$ = I^{m,n}_{p_i,q_i,r}\left[z\,\middle|\,\begin{matrix}(a_1,A_1)_{1,n},(a_{ji},A_{ji})_{n+1,p_1}\\ (b_j,B_j)_{1,m},(b_{ji},B_{ji})_{m+1,q_1}\end{matrix}\right] \qquad (1.11) $$

(iv) Further setting $\rho_i = 1\ and\ r = 1$ in (1.1) and (1.3) reduces into the incomplete H-function introduced by Srivastava et. al.[13] :

$$ {}^{(\Gamma)}(\aleph)^{m,n}_{p_i,q_i,1,1}\left[z\,\middle|\,\begin{matrix}(a_1,A_1,y),(a_j,A_j)_{2,n},(a_{ji},A_{ji})_{n+1,p_1}\\ (b_j,B_j)_{1,m},[1(b_{ji},B_{ji})]_{m+1,q_1}\end{matrix}\right] $$

$$ = \Gamma^{m,n}_{p,q}\left[z\,\middle|\,\begin{matrix}(a_1,A_1,y),(a_j,A_j)_{2,p}\\ (b_j,B_j)_{1,q}\end{matrix}\right] \qquad (1.12) $$

$$^{(\Gamma)}(\aleph)^{m,n}_{p_i,q_i,1,1}\left[z\left|\begin{matrix}(a_1,A_1,y),(a_j,A_j)_{2,n},[1(a_{ji},A_{ji})]_{n+1,p_1}\\(b_j,B_j)_{1,m},[1(b_{ji},B_{ji})]_{m+1,q_1}\end{matrix}\right.\right]=\gamma^{m,n}_{p,q}\left[z\left|\begin{matrix}(a_1,A_1,y)\ \ ,(a_j,A_j)_{2,p}\\(b_j,B_j)_{1,q}\end{matrix}\right.\right] \quad (1.13)$$

A complete details of Incomplete H-function can be found in article [10],see also [2].

(v) Again setting y=0 in (1.12), incomplete ℵ-function deduce into the Fox H-function [12] as follows:

$$^{(\Gamma)}(\aleph)^{m,n}_{p_i,q_i,1,1}\left[z\left|\begin{matrix}(a_1,A_{1,0}),(a_j,A_j)_{2,n},[1(a_{ji},A_{ji})]_{n+1,p_1}\\(b_j,B_j)_{1,m},[1(b_{ji},B_{ji})]_{m+1,q_1}\end{matrix}\right.\right]=H^{m,n}_{p,q}\left[z\left|\begin{matrix}(a_j,A_j)_{1,p}\\(b_j,B_j)_{1,q}\end{matrix}\right.\right] \quad (1.14)$$

The well-known Bessel-Maitland function $J^{\alpha}_{\beta}(z)$ is introduced and define by Edward Maitland Wright (see,Marichev [6])and which is given as:

$$J^{\alpha}_{\beta}(z)=\sum_{m=0}^{\infty}\frac{(-z)^m}{m!\ \Gamma(\alpha m+\beta+1)},where\ \ Re(\alpha)\geq -1,Re(\beta)\geq 0,z\epsilon\mathbb{C}. \quad (1.15)$$

Singh et al. [11] defined and studied the more generalized Bessel Maitland function as follows:

$$J^{\alpha,\gamma}_{\beta,q}(z)=\sum_{m=0}^{\infty}\frac{(\gamma)_{qm}(-z)^m}{\prod_{j=1}^{n}\Gamma\,(\alpha m+\beta+1)m!} \quad (1.16)$$

($\Re(\alpha)>\max\{0,q-1\},\Re(\beta)>-1,\Re(\gamma)>0,x\epsilon\mathbb{C}\ and\ \ q\epsilon(0,1)\cup\mathbb{N}$).

Saxena and Nishimoto [10] defined and studied the following multi-index Mittag-Leffler function which is equivalent to multi-index Bessel-Maitland function:

$$J^{(\alpha_i),\gamma}_{(\beta_j),q}(x)=\sum_{m=0}^{\infty}\frac{(\gamma)_{qm}(-x)^m}{\prod_{j=1}^{n}\Gamma\left(\alpha_j m+\beta_j+1\right)m!} \quad (1.17)$$

where $\alpha_j,\beta_j,\gamma\ and\ z\epsilon\ \mathbb{C}\ such\ that$

$$\Re(\gamma)>0,\Re(\beta_j)>-1,\sum_{j=1}^{n}\Re(\alpha_j)>max\{0,\Re(q)-1\},q\in(0,1)\ \cup\mathbb{N}$$

where $(\gamma)_{qm}$ is the Pochhammer symbol, which can be written in terms of gamma function as

$$(\gamma)_{qm}=\frac{\Gamma(\gamma+mq)}{\Gamma(\gamma)} \quad (1.18)$$

The well-known orthogonal Jacobi polynomial of degree ν is defined in the following manner (for details, see, [14, p. 59, Eq. (4.1.3)] and [15, p.35, Eq. (34)],[15]):

$$p^{(\alpha,\beta)}_{V}(x)=\frac{(1+\alpha)_V}{V!}\sum_{R=0}^{V}\frac{(-V)_R(1+\alpha+\beta+V)_R}{(1+\alpha)_R\ R!}\left(\frac{1-x}{2}\right)^R \quad (1.19)$$

The following integral formulas are necessary for it.

$$\int_0^1 x^{\sigma}(1-x^2)^{-\mu/2}\ \ p^{\mu}_{v}(x)dx=\frac{2^{\mu-1}\Gamma\left(\frac{1}{2}+\frac{\sigma}{2}\right)\left(1+\frac{\sigma}{2}\right)}{\Gamma\left(1+\frac{\sigma}{2}-\frac{v}{2}-\frac{\mu}{2}\right)\Gamma\left(\frac{\sigma}{2}+\frac{v}{2}-\frac{\mu}{2}+\frac{3}{2}\right)} \quad (1.20)$$

provided that $\Re(\mu)<1,\Re(\sigma)>-1$

$$\int_0^\infty x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda} dz = 2\lambda\, a^{-\lambda}\,\frac{a^{\mu}\Gamma(2\mu)\Gamma(\lambda-\mu)}{2^{\mu}\Gamma(1+\lambda+\mu)} \tag{1.21}$$

provided that $0 < \Re(\mu) < \Re(\lambda)$.

The formula in (1.21) is known as Oberthettinger [4],[5].

2. A set of Integrals

In this part, we derive a number of finite integral formulas involving the Legendre function, incomplete Aleph function, Multi-index Bessel Maitland function and Jacobi polynomial

Theorem .1: If$\alpha_j, \beta_j, \gamma$ *and* $z \in \mathbb{C} < 1$**,**$\Re(\mu) < 1$,$\Re(\sigma) < -$,$\Re(\gamma) > 0$; $\Re(\beta_j) > -1, (I = 1, \ldots, n)$, and $\sum_{j=1}^{n} \Re(\alpha_j) > max\{0, \Re(q) - 1\}, q \in (0,1) \cup \mathbb{N}$, then the following integral formulas hold under the condition stated:

$$\int_0^1 z^{\sigma}(1-z^2)^{-\mu/2}\; P_\nu^\mu(z)\; J_{(\beta_j),q}^{(\alpha_i),\gamma}\left(\frac{(az)}{(1-z^2)^{\tau/2}}\right) dz = \frac{2^{\mu-1}}{\Gamma\gamma}$$

$$\times\ {}_3\psi_{n+2}\left[\begin{matrix}\left(\frac{1+\sigma}{2},\frac{1}{2}\right),(\gamma,q),\left(1+\frac{\sigma}{2},\frac{1}{2}\right)\\ (\beta_1+1,\alpha_1),\ldots\ldots..,(\beta_n+1,\alpha_n),\left(1+\frac{\sigma-\nu-\mu}{2},\frac{(1-\tau)}{2}\right)\\ ;\\ \Gamma\left(\frac{\sigma+\nu-\mu+3}{2},\frac{(1-\tau)}{2}\right);\ -2^{\tau}a\end{matrix}\right] \tag{2.1}$$

Proof: To prove the assertion(2.1), utilising (1.17) in the left hand side and rearranging the order of summation and integration, which is guaranteed by the convergence criteria and we will get say (I$_1$):

$$I_1 = \sum_{m=0}^{\infty}\frac{(\gamma)_{qm}a^m}{\prod_{j=1}^{n}\Gamma(\alpha_j m+\beta_j+1)m!}\int_0^1 z^{\sigma+m-1}(1-z^2)^{-\mu/2-\tau m/2}\; P_\nu^\mu(z)dz$$

Next, with the help of (1.20)

$$I_1 = \frac{2^{\mu-1}}{\Gamma(\gamma)}\sum_{m=0}^{\infty}\frac{2^{\tau m}a^m\Gamma(\gamma+qm)\Gamma\left(\frac{1+\sigma+m}{2}\right)\Gamma\left(1+\frac{\sigma+m}{2}\right)}{\prod_{j=1}^{n}\Gamma(\alpha_j m+\beta_j+1)\; m!\;\Gamma\left(1+\frac{\sigma+m-\nu-\mu-\tau m}{2}\right)\Gamma\left(\frac{\sigma+m+\nu-\mu-\tau m+3}{2}\right)}$$

Next we using result given by [7, eq (1.9)], we acquire our result.

Theorem 2.2: If$\alpha_j, \beta_j, \gamma$ *and* $z \in \mathbb{C} < 1$,$\Re(\mu) < 1$,$\Re(\sigma) < -$,$\Re(\gamma) > 0$; $\Re(\beta_j) > -1, (I = 1, \ldots, n)$, and $\sum_{j=1}^{n} \Re(\alpha_j) > max\{0, \Re(q) - 1\}, q \in (0,1) \cup \mathbb{N}$, then the following integral formulas hold under the condition stated:

$$\int_0^1 z^{\sigma}(1-z^2)^{-\mu/2}\; P_\nu^\mu(z)\; J_{(\beta_j),q}^{(\alpha_i),\gamma}(az^{\sigma_1})P_n^{(\alpha,\beta)}[y(1-z^{\sigma_2})]dz$$

$$= \frac{2^{\mu-1}\Gamma(1+\alpha+V)}{V!\,\Gamma\gamma}\sum_{R=0}^{V}\frac{(-V)_R\Gamma(1+\alpha+\beta+V)_R}{R!\,2^R\Gamma(1+\alpha+R)}$$

$${}_3\psi_{n+2}\left[\begin{matrix}\left(\frac{1+\sigma+\sigma_2R}{2},\frac{\sigma_1}{2}\right),(\gamma,q),\left(1+\frac{1+\sigma+\sigma_2R}{2},\frac{\sigma_1}{2}\right) & ;\\ (\beta_1+1,\alpha_1),\ldots\ldots..,(\beta_n+1,\alpha_n),\left(1+\frac{\sigma+\sigma_2R-\nu-\mu}{2},\frac{\sigma_1}{2}\right),\Gamma\left(\frac{\sigma+\sigma_2R+\nu-\mu+3}{2},\frac{\sigma_1}{2}\right); & \end{matrix} -a\right] \tag{2.2}$$

Proof: Using (1.17) & (1.19) in the left hand side and interchanging the order of summation and integration which is garranteed under the convergence conditions

$$= \sum_{m=0}^{\infty} \frac{(\gamma)_{qm} a^m}{\prod_{j=1}^{n} \Gamma\left(\alpha_j m + \beta_j + 1\right) m!} \frac{(1+\alpha_1)_V}{V!} \sum_{R=0}^{V} \frac{(-V)_R (1+\alpha+\beta+V)_R y^R}{(1+\alpha)_R R!\, 2^R}$$

$$\times \int_0^1 z^{\sigma+\sigma_1 m+\sigma_2 R-1} (1-z^2)^{-\mu/2}\, P_\nu^\mu(z) dz$$

Now, applying (1.20) we achieve

$$= \frac{2^{\mu-1}\Gamma(1+\alpha+V)}{V!\,\Gamma(\gamma)} \sum_{R=0}^{V} \frac{(-V)_R (1+\alpha+\beta+V)_R y^R}{R!\, 2^R \Gamma(1+\alpha+R)}$$

$$\times \sum_{m=0}^{\infty} \frac{a^m \Gamma(\gamma+qm) \Gamma\left(\frac{1+\sigma+\sigma_1 m+\sigma_2 R}{2}\right) \Gamma\left(1+\frac{\sigma+\sigma_1 m+\sigma_2 R}{2}\right)}{\prod_{j=1}^{n} \Gamma\left(\alpha_j m+\beta_j+1\right) \Gamma\left(1+\frac{\sigma+\sigma_1 m+\sigma_2 R-\nu-\mu}{2}\right) \Gamma\left(\frac{\sigma+\sigma_1 m+\sigma_2 R+\nu-\mu+3}{2}\right) m!}$$

After that, we apply khan’s result given in [7] to obtain our own.

Theorem 2.3: **Let** $y \geq 0\ and\ \Re(\vartheta) > 0, \Omega > 0, |\arg z| < \frac{\pi\Omega}{2}, \Re(\mu) < 1, \Re(\sigma) < -1$ $j = 1,2,\dots,n;\ \Re(\gamma) > 0, \Re(\beta_j) > 1, \Re[\sum_{j=1}^{n} \alpha_j] > \max\{0, q-1\}$.

$$\int_0^1 z^\sigma (1-z^2)^{-\mu/2}\ P_\nu^\mu(z)\ {}^{(\Gamma)}\aleph_{p_i q_{i,\rho_{i,,}},r}^{m,n} \left(\frac{xz^2}{\sqrt{(1-z^2)}}\right)\ J_{(\beta_j),q}^{(\alpha_i),\gamma} \left(\frac{(a)}{(1-z^2)^{1/2}}\right) dz$$

$$= 2^{\mu-1} \sum_{m=0}^{\infty} \frac{\Gamma(\gamma)_{qm} (2a)^m}{\prod_{j=1}^{n} \Gamma\left(\alpha_j m+\beta_j+1\right) m\,!}\ {}^{(\Gamma)}(\aleph)_{p_{i+2},q_{i+2},\rho_i,r}^{m,n+2} \left[2x \middle| \begin{matrix} (a_1, A_1, y), \left(a_j, A_j\right)_{2,n}, \left(\frac{1+\sigma}{2}, 1\right), \\ \left(b_j, B_j\right)_{1,m}, \left(1+\frac{\sigma-\nu-\mu}{2}, \frac{1}{2}\right) \end{matrix} \right.$$

$$\left. \begin{matrix} \left(1+\frac{\sigma}{2}, 1\right), [\rho_j\left(a_{ji}, A_{ji}\right)]_{n+1,p_1} \\ \left(\frac{\sigma+\nu-\mu+3}{2}, \frac{1}{2}\right), [\rho_j\left(b_{ji}, B_{ji}\right)]_{m+1,q_1} \\ , \end{matrix} \right] \qquad (2.3)$$

Proof: To prove theorem (2.3) Using the (1.3) and (1.17) in the left hand side of the above result and interchanging the order of summation and integration, which is also satisfies under the convergence condition then we will get :

$$= \sum_{m=0}^{\infty} \frac{(\gamma)_{qm} a^m}{\prod_{j=1}^{n} \Gamma\left(\alpha_j m+\beta_j+1\right) m!} \frac{1}{2\pi i} \int_c\ K(\xi, y)(x)^{-\xi}$$

$$\times \int_0^1 z^{\sigma+2\xi-1} (1-z^2)^{-\frac{(\mu+m+\xi)}{2}}\ P_\nu^\mu(z)\ d\xi\ dz$$

Now, we applying (1.20) and using the result given by khan, so we get

$$= 2^{\mu-1} \sum_{m=0}^{\infty} \frac{(2a)^m (\gamma)_{qm}}{\prod_{j=1}^{n} \Gamma\left(\alpha_j m+\beta_j+1\right) m!} \frac{1}{2\pi i} \int_c\ K(\xi, y)(x)^{-\xi} 2^\xi$$

$$\times\ \frac{\Gamma\left(\frac{1+\sigma}{2}+\xi\right) \Gamma\left(1+\frac{\sigma}{2}+\xi\right)}{\Gamma\left(1+\frac{\sigma-\nu-\mu-m+\xi}{2}\right) \Gamma\left(\frac{\sigma+\nu-\mu+3+\xi}{2}\right)}\ d\xi$$

Next we use the (1.4) in the above integral then we acquire our result after little simplification.

Theorem 2.4: ***T****he under-mentioned result holds true under specified conditions*

$$\int_0^1 z^{\sigma}(1-z^2)^{-\mu/2}\ P_\nu^\mu(z)\ {}^{(\gamma)}\aleph_{p_i q_{i,\rho_{i,,}},r}^{m,n}\left(\frac{xz^2}{\sqrt{(1-z^2)}}\right)\ J_{(\beta_j),q}^{(\alpha_i),\gamma}\left(\frac{(a)}{(1-z^2)^{1/2}}\right)$$

$$=\frac{2^{\mu-1}}{\Gamma(\gamma)}\sum_{m=0}^{\infty}\frac{\Gamma(\gamma)_{qm}(2a)^m}{\prod_{j=1}^{n}\Gamma\big(\alpha_j m+\beta_j+1\big)m\,!}$$

$${}^{(\gamma)}(\aleph)_{p_{i+2},q_{i+2},\rho_i,r}^{m,n+2}\left[2x\left|\begin{matrix}(a_1,A_1,y),\big(a_j,A_j\big)_{2,n},\left(\frac{1+\sigma}{2},1\right),\left(1+\frac{\sigma}{2},1\right),[\rho_j\big(a_{ji},A_{ji}\big)]_{n+1,p_1}\\ \big(b_j,B_j\big)_{1,m},\left(1+\frac{\sigma-\nu-\mu-m}{2},\frac{1}{2}\right),\left(\frac{\sigma+\nu-\mu+3}{2},\frac{1}{2}\right),[\rho_j\big(b_{ji},B_{ji}\big)]_{m+1,q_1}\end{matrix}\right.\right]\quad(2.4)$$

$$where\ \ y\geq 0\ and\ \ \Re(\vartheta)>0,\Omega>0,k>1,|\arg z|<\frac{\pi\Omega}{2},\Re(\mu)<1\,,\Re(\sigma)<-1$$

$$j=1,2,\dots,n;\ \Re(\gamma)>0,\Re\big(\beta_j\big)>1,\Re[\textstyle\sum_{j=1}^{n}\alpha_j]>\max\{0,q-1\}\,,q\epsilon(0,1)$$

Proof: To demonstrate (2.4) is on the similar lines as of Theorem (2.3).

Theorem 2.5: The result below is verifiable under specified conditions

$$\int_0^\infty x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda}\ J_{(\beta_j),q}^{(\alpha_i),\gamma}\left(\frac{Z}{\left(x+a+\sqrt{x^2+2ax}\right)^{\lambda_1}}\right)$$

$$X\ {}^{(\Gamma)}\aleph_{p_i q_{i,\rho_{i,,}},r}^{m,n}\left(\frac{zx^{\mu_1}}{\left(x+a+\sqrt{x^2+2ax}\right)^{\lambda_2}}\right)dx=\sum_{m=0}^{\infty}\frac{\Gamma(\gamma)_{qm}(2a)^m}{\prod_{j=1}^{n}\Gamma\big(\alpha_j m+\beta_j+1\big)m\,!}$$

$${}^{(\Gamma)}(\aleph)_{p_{i+2},q_{i+2},\rho_i,r}^{m,n+2}\left[2x\left|\begin{matrix}(a_1,A_1,y),\big(a_j,A_j\big)_{2,n},\left(\frac{1+\sigma}{2},1\right),\left(1+\frac{\sigma}{2},1\right),\left[\rho_j\big(a_{ji},A_{ji}\big)\right]_{n+1,p_1}\\ \big(b_j,B_j\big)_{1,m},\left(1+\frac{\sigma-\nu-\mu-m}{2},\frac{1}{2}\right),\left(\frac{\sigma+\nu-\mu+3}{2},\frac{1}{2}\right),\left[\rho_j\big(b_{ji},B_{ji}\big)\right]_{m+1,q_1}\end{matrix}\right.\right](2.5)$$

$$y\geq 0\ and\ \ \Re(\vartheta)0,\Omega>0,k>1,|\arg z|<\frac{\pi\Omega}{2},\Re(\mu)<1\,,\Re(\sigma)<-1$$

$$j=1,2,\dots,n;\ \Re(\gamma)>0,\Re\big(\beta_j\big)>1,\Re[\textstyle\sum_{j=1}^{n}\alpha_j]>\max\{0,q-1\}\,,q\epsilon(0,1)$$

Proof*:* To prove the assertion (2.5), using (1.3)and (1.17) in the left hand side and interchanging the order of summation and integration which is guaranteed under the convergence conditions, we get :

$$=\frac{1}{2\pi i}\int_c K(\xi,y)(z)^{-\xi}\sum_{m=0}^{\infty}\frac{(\gamma)_{qm}Z^m}{\prod_{j=1}^{n}\Gamma\big(\alpha_j m+\beta_j+1\big)m!}$$

$$\times\int_0^\infty x^{\mu-\mu_1\xi-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda-\lambda_1 m-\lambda_2\xi}dx\,d\xi.$$

To solve the above internal integral, we will using the (1.20), then we get

$$=\frac{1}{2\pi i}\int_c K(\xi,y)(z)^{-\xi}\,2(\lambda+\lambda_1 m+\lambda_2\xi)\,a^{-\lambda-\lambda_1 m-\lambda_2\xi}\sum_{m=0}^{\infty}\frac{(2a)^m(\gamma)_{qm}}{\prod_{j=1}^{n}\Gamma\big(\alpha_j m+\beta_j+1\big)m!}$$

$$X\ \frac{a^{\mu-\mu_1\xi}\,\Gamma\big(2(\mu-\mu_1\xi)\big)\Gamma(\lambda+\lambda_1 m+\lambda_2\xi-\mu+\mu_1\xi\ )}{2^{\mu-\mu_1\xi}\Gamma(1+\lambda+\lambda_1 m+\lambda_2\xi+\mu-\mu_1\xi)}\ d\xi.$$

Now,Using (1.4),we acquire the right –hand side in Theorem 2.5 after little simplification.

Theorem 2.6: The under-mentioned result holds true under specified conditions

$$\int_0^\infty x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda} J_{(\beta_j),q}^{(\alpha_i),\gamma}\left(\frac{Z}{\left(x+a+\sqrt{x^2+2ax}\right)^{\lambda_1}}\right)$$

$$X\ {}^{(\gamma)}\aleph_{p_i q_{i,\rho_{i,,},r}}^{m,n}\left(\frac{zx^{\mu_1}}{\left(x+a+\sqrt{x^2+2ax}\right)^{\lambda_2}}\right)dx$$

$$=\sum_{m=0}^{\infty}\frac{\Gamma(\gamma)_{qm}(2a)^m}{\prod_{j=1}^{n}\Gamma\left(\alpha_j m+\beta_j+1\right)m\,!}$$

$${}^{(\gamma)}(\aleph)_{p_{i+2},q_{i+2},\rho_i,r}^{m,n+2}\left[2x\left|\begin{matrix}(a_1,A_1,y),\left(a_j,A_j\right)_{2,n},\left(\frac{1+\sigma}{2},1\right),\left(1+\frac{\sigma}{2},1\right),[\rho_j\left(a_{ji},A_{ji}\right)]_{n+1,p_1}\\ \left(b_j,B_j\right)_{1,m},\left(1+\frac{\sigma-\nu-\mu-m}{2},\frac{1}{2}\right),\left(\frac{\sigma+\nu-\mu+3}{2},\frac{1}{2}\right),[\rho_j\left(b_{ji},B_{ji}\right)]_{m+1,q_1}\end{matrix}\right.\right] \tag{2.6}$$

$\Re(\vartheta)0,\Omega>0,k>1,|\arg z|<\frac{\pi\Omega}{2},\Re(\mu)<1\,,\Re(\sigma)<-1,j=1,2,\dots,n;\Re(\gamma)>0$
$,\Re\left(\beta_j\right)>1,\Re[\sum_{j=1}^{n}\alpha_j]>\max\{0,q-1\}\,,q\epsilon(0,1),then\ \ for\ y\geq 0$

Proof: The proof of Theorem 2.6 is on similar lines as the Theorem 2.5.

Theorem 2.7: The under-mentioned result holds true under specified conditions

$$\int_0^\infty x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda} J_{(\beta_j),q}^{(\alpha_i),\gamma}(x^{\mu_1}y)\,P_n^{(\alpha,\beta)}[y(1-x^{\mu_2})]dz$$

$$=\lambda\frac{a^{\mu-\lambda}\Gamma(1+\alpha+V)}{2^{\mu-1}\mathrm{V}!\,\Gamma\gamma}\sum_{R=0}^{V}\frac{(-V)_R\Gamma(1+\alpha+\beta+V)_R}{(2a)^{\mu_2 R}R!\,\Gamma(1+\alpha+R)}$$

$${}_3\psi_{n+2}\left[\begin{matrix}(2\mu+2\mu_2 R,2\mu_1),(\gamma,q),(\lambda-\mu-\mu_2 R,-\mu_1) & ;\\ (\beta_1+1,\alpha_1),\dots\dots\dots,(\beta_n+1,\alpha_n),(1+\lambda+\mu+\mu_2 R,-\mu_1); & \end{matrix}-2^{\mu_1}y\right]$$

(2.7)

$\Re(\vartheta)0,\Omega>0,k>1,|\arg z|<\frac{\pi\Omega}{2},\Re(\mu)<1\,,\Re(\sigma)<-1,j=1,2,\dots,n;\Re(\gamma)>0$
$,\Re\left(\beta_j\right)>1,\Re[\sum_{j=1}^{n}\alpha_j]>\max\{0,q-1\}\,,q\epsilon(0,1),then\ \ for\ y\geq 0$

Proof: Using (1.17) & (1.19) in the left hand side and interchanging the order of summation and integration which is guaranteed under the convergence conditions

$$=\sum_{m=0}^{\infty}\frac{(\gamma)_{qm}a^m}{\prod_{j=1}^{n}\Gamma\left(\alpha_j m+\beta_j+1\right)m!}\frac{(1+\alpha_1)_V}{V!}\sum_{R=0}^{V}\frac{(-V)_R(1+\alpha+\beta+V)_R y^R}{(1+\alpha)_R R!\,2^R}$$

$$\int_0^\infty x^{\mu-\mu_1 m-\mu_2 R-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda}dx$$

Now, applying (1.20), after that, we apply result given by [7].Then we obtain our result (2.7) after little simplification.

3. In the aforementioned theorems, the Fox-Wright function is transformed into the Fox-H function.

We will use in this section to carry out the Fox-Wright function to Fox-H function transformation in Theorem 1.

Corollary 3.1 Variation of Theorem 2.1

$$\int_0^1 z^{\sigma}(1-z^2)^{-\mu/2}\ P_{\nu}^{\mu}(z) J_{(\beta_j),q}^{(\alpha_i),\gamma}\left(\frac{(az)}{(1-z^2)^{\frac{\tau}{2}}}\right)\ dz$$

$$= \frac{2^{\mu-1}}{\Gamma\gamma\Gamma\left(1+\frac{\sigma-\nu-\mu}{2}\right)\Gamma\left(\frac{\sigma+\nu-\mu+3}{2}\right)}$$

$$\times\ H_{3,n+1}^{1,3}\left[\begin{matrix}\left(\frac{1-\sigma}{2},\frac{1}{2}\right),(1-\gamma,q),\left(-\frac{\sigma}{2},\frac{1}{2}\right) & ; \\ (0,1),(-\beta_1,\alpha_1),\dots\dots\dots,(-\beta_n,\alpha_n); & \end{matrix} -2^{\tau}a\right] \qquad (3.1)$$

4. Particular Cases

In this part, we've provided a few intriguing applications of our key discovery.

Corollary 4.1: By setting j=1 and putting $\alpha_1=\alpha,\beta_1=\beta\ in\ the\ theorem$ 1.

$$\int_0^1 z^{\sigma}(1-z^2)^{-\mu/2}\ P_{\nu}^{\mu}(z) J_{\beta,q}^{\alpha,\gamma}\left(\frac{(az)}{(1-z^2)^{\tau/2}}\right)dz = \frac{2^{\mu-1}}{\Gamma\gamma\Gamma\left(1+\frac{\sigma-\nu-\mu}{2}\right)}$$

$$\times\frac{1}{\Gamma\left(\frac{\sigma+\nu-\mu+3}{2}\right)}\ {}_3\psi_1\left[\begin{matrix}\left(\frac{1+\sigma}{2},\frac{1}{2}\right),(\gamma,q),\left(1+\frac{\sigma}{2},\frac{1}{2}\right) & ; \\ (\beta+1,\alpha) & \end{matrix} -2^{\tau}a\right] \qquad (4.1)$$

Corollary 4.2: If we take j = 1 and α₁ = α, β₁ =β, q = 0 Theorem 2.1, then

$$\int_0^1 z^{\sigma}(1-z^2)^{-\mu/2}\ P_{\nu}^{\mu}(z) J_{\beta}^{\alpha}\left(\frac{(az)}{(1-z^2)^{\tau/2}}\right)dz = \frac{2^{\mu-1}}{\Gamma\left(1+\frac{\sigma-\nu-\mu}{2}\right)\Gamma\left(\frac{\sigma+\nu-\mu+3}{2}\right)}$$

$$\times\ {}_2\psi_1\left[\begin{matrix}\left(\frac{1+\sigma}{2},\frac{1}{2}\right),\left(1+\frac{\sigma}{2},\frac{1}{2}\right) & ; \\ (\beta+1,\alpha) & \end{matrix} -2^{\tau}a\right] \qquad (4.2)$$

Corollary 4.3: If $y\geq 0\ and\ \Re(\vartheta)>0,\Omega>0,2(m+n).p+qh>0,|\arg z|<\pi\left(m+n-\frac{p}{2}-\frac{q}{2}\right),\Re(\mu)<1,$

then the following integral formula involving of the incomplete Meijer ${}^{(\Gamma)}$G-function as given below:.

$$\int_0^1 x^{\sigma}\left(1-x^2\right)^{-\mu/2}\ p_{\upsilon}^{\mu}(x)\ \ {}^{(\Gamma)}G_{p,q}^{m,n}\left[zx^2\left|\begin{matrix}(a_j,y),(a_j)_{2,p} \\ (b_j)_{1,q}\end{matrix}\right.\right]dx$$

$$= 2^{\mu-1}\,{}^{(\Gamma)}G_{p+2,q+2}^{m,n+2}\left[z\left|\begin{matrix}(a_1;y),\left(\frac{1-\sigma}{2}\right),\left(-\frac{\sigma}{2}\right)(a_j)_{2,p} \\ \left(\frac{\upsilon+\mu-\sigma}{2}\right),\left(\frac{\mu-\upsilon-\sigma-1}{2}\right)(b_j)_{1,q}\end{matrix}\right.\right] \qquad (4.3)$$

Proof: Taking i = 1,…….,m =1, A_i=1 and B_j = 1 in the result (13),we get the desire result.

Corollary4.4: *If* $y \geq 0$ *and* $\Re(\vartheta) > 0, \Omega > 0, 2(m+n).p + qh > 0, |\arg z| < \pi\left(m + n - \frac{p}{2} - \frac{q}{2}\right),$ $\Re(\mu) < 1,$ then the following integral formula involving of the incomplete Fox-Wright ${}_p\psi_q^{(\Gamma)}$-function [9] as follows:.

$$\int_0^1 x^{\sigma}\left(1-x^2\right)^{-\mu/2} \quad p_{\upsilon}^{\mu}(x) \quad {}_p\psi_q^{(\Gamma)}\left[zx^k \left| \begin{matrix} (a_j, y), (a_j)_{2,p} \\ (b_j, B_j)_{1,q} \end{matrix} \right.\right]dx$$

$$= 2^{\mu-1}\, {}_{p+2}\psi_{,q+2}^{(\Gamma)}\left[z \left| \begin{matrix} (a_1, A_1; y), \left(\frac{1-\sigma}{2}, h\right), \left(-\frac{\sigma}{2}, h\right)(a_j, A_j)_{2,p} \\ \left(\frac{\upsilon+\mu-\sigma}{2}, h\right), \left(\frac{\mu-\upsilon-\sigma-1}{2}, h\right)(b_j, B_j)_{1,q} \end{matrix} \right.\right] \tag{4.4}$$

Proof: Taking m=1 and n = p, replacing q with q=1, and the suitable adjustment of the parameters the incomplete H-function reduces to incomplete Fox-Wright ${}_p\psi_q^{(\Gamma)}$-function [11] also see [3,5,7].

Remark 1: If the incomplete Aleph function is unity in theorem 4 and 5, then the result is that recorded by Khan and Khan [7].

The many types of integral formulas incorporating the various kinds of special functions have been the subject of numerous research articles written by various writers. Regarding this, we developed certain integral formulas related to the Legendre function, multiindex Bessel's Maitland function, and incomplete Aleph function. Additionally, we present several intriguing, supposedly novel special examples of these integrals.

5. References

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