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Generalized On Complexity And Hamiltonian Cycles In Graphs And Applications

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Abstract
In this work, In a cubic graph, an edge's number of Hamiltonian cycles is even. This theorem results in an algorithm that provides an exponential lower bound for a given Hamiltonian cycle in such a graph. Complexity of regular graphs or the Hamiltonian cycle in them. Finding the Hamiltonian cycle (or path) of a 3- normal graph was shown to be NP-complete. We show that the task of determining if a k-graph has a Hamiltonian cycle (or path) is NP-complete for every set k≥6 and 3. Determining if a planar k-regular graph has a Hamiltonian cycle (or path) has been shown to be NP-complete for k = 3. We demonstrate that the problem is NP-complete for k = 4 and k = 5

1.Introduction:

Study the complexity of search problems for total functions, in which the existence of a solution is guaranteed via simple combinatorial arguments, but no efficient algorithmic solutions are known. See also [3, 5, 1] for other related works. One of the problems considered in [4] is the following: given a cubic graph G, and a Hamiltonian cycle C in G. When the graph is not constrained to be planar, for 4-regular graph, the problem was conjectured to be NP-complete. In this paper, we first prove that for any fixed k≥3, deciding whether a k- regular graph has a Hamiltonian cycle (or path) is a NP-complete problem. Secondly, we will return to the subproblem of planar k-regular graph. In this case, the problem is obviously polynomial for k ≥ 6 since a planar graph cannot have a vertex with degree larger. For Hamiltonian cycles, we present a novel measure of the complexity.

Review of literature

1. Itai et al. in 1982 [8]The Hamiltonian path and cycle problems have been studied in detail in the context of graphs
2. Later Umans and Lenhart [9] it is NP-complete to decide whether a graph has a Hamiltonian path or a Hamiltonian cycle. They also gave necessary and sufficient conditions for a rectangular graph to have a Hamiltonian cycle. They left the problem of deciding whether a Hamiltonian cycle exists in a graph open.
3. Garey et al. [2] proved deciding whether a 3- planar graph has a Hamiltonian cycle is a NP-complete problem

4. Afrati [1] a polynomial time algorithm to find a Hamiltonian cycle (if it exists) in a graph. gave a linear time algorithm for finding Hamiltonian cycles in restricted.
5. Cho and Zelikovsky [4] studied spanning closed trails containing all the vertices of a graph..
6. Arkin et al. [2] studied the existence of Hamiltonian cycles in graphs and proved several complexity results.
7. Garcyel.al[2] provided deciding whether a graph has a Hamiltonian cycle is a NP-complete.
8. Papadimotriow [3] proposed to study complexity of search for total functions.

2 Preliminaries

In this section, we introduce Hamiltonian cycles /path on graphs, and a complexity measure for such cycles that we call the complexity.

Definition 2.1

Graph is a diagram showing the relation between variable quantities, typically of two variables, each measured along one of a pair of axes at right angles.

Remark2.2:

The term Complexity has two distinct usages, which may be categorized simply as either a quality or a quantity. We often speak of complex systems as being a particular class of systems that are difficult to study using traditional analytic techniques. We have in mind that biological organisms and ecosystems are complex, yet systems like a pendulum, or a lever are simple. Complexity as a quality is therefore what makes the systems complex

Definition 2.3.

The complexity of a graph generally refers to the difficulty in understanding its structural and algorithmic properties. It encompasses various aspects such as size, connectivity, density, and the presence of specific patterns or structures within the graph.

Definition 2.4.

A Hamiltonian cycle, also called a Hamiltonian circuit, Hamilton cycle, or Hamilton circuit, is a graph cycle (i.e., closed loop) through a graph that visits each node exactly once. A graph possessing a Hamiltonian cycle is said to be a Hamiltonian graph. and a path that uses every vertex in a graph exactly once is called a Hamilton path

Dirac's Theorem 2.5

If G is a simple graph with n vertices, where $n \geq 3$ If $deg(v) \geq \{n\}/\{2\}$ for each vertex v, then the graph G is Hamiltonian graph.

Ore's Theorem 2.6

If G is a simple graph with n vertices, where $n \geq 2$ if $deg(x) + deg(y) \geq n$ for each pair of non-adjacent vertices x and y, then the graph G is Hamiltonian graph.

Definition 2.7:

Transpose of a directed graph G is another directed graph on the same set of vertices with all of the edges reversed compared to the orientation of the corresponding edges in G. That is, if G contains an edge (u, v) then the converse/transpose/reverse of G contains an edge (v, u) and vice versa.

3.Main Results Theorem 3.1:

Every graph G has an even number of Hamiltonian cycles, containing a given edge.

Proof: by using Handshaking lemma

To prove that every graph *G* has an even number of Hamiltonian cycles containing a given edge, "Handshaking Lemma."

the sum of the degrees of all vertices is twice the number of edges. Mathematically, it can be expressed as: $2|E| = \Sigma v \in V \deg(v)$

where $|E|$ is the number of edges, V is the set of vertices, and $\deg(v)$ denotes the degree of vertex v

1. Let *G* be a graph containing a given edge *e*.
2. Suppose there are *n* Hamiltonian cycles containing *e*.
3. In each Hamiltonian cycle containing *e*, each vertex is visited exactly twice (once before traversing *e* and once after traversing *e*) except for the endpoints of *e*, which are visited three times.
4. Considering the Handshaking Lemma, the total number of visits to all vertices in all Hamiltonian cycles containing *e* must be an even number.
5. Since each Hamiltonian cycle contributes an even number of visits to the vertices, and the total number of visits is even, the total number of Hamiltonian cycles containing *e* must be even.

Hence, every graph *G* has an even number of Hamiltonian cycles containing a given edge *e*.

This proof relies on the Handshaking Lemma and the observation that each Hamiltonian cycle contributes an even number of visits to the vertices, which leads to the evenness of the total count of Hamiltonian cycles containing the given edge.

pseudo-code representation of the algorithm:

```
Algorithm CountHamiltonianCycles(G, e):
    count = 0

    // Step 1: Identify the vertices incident to the given edge e
    v1, v2 = vertices_incident_to_edge(e)

    // Step 2: Generate all Hamiltonian cycles in graph G
    hamiltonian_cycles = GenerateAllHamiltonianCycles(G)

    // Step 3: Filter cycles containing the given edge e
    for each cycle in hamiltonian_cycles:
        if cycle_contains_edge(cycle, e, v1, v2):
            count = count + 1

    // Step 4: Output the count of Hamiltonian cycles containing the given edge
    return count
```

Theorem 3.2.

For any n≥1, there exists a graph Gn with 16n+2 vertices, an edge e of G, and an initial Hamiltonian cycle C in G containing e, for which makes 2^n steps.

Consider a cycle graph with 16*n*vertices, denoted as C_{16n}. Now, addtwo additional vertices, *u*and *v*, to the graph. (four vertices , denotedby A,B,C,and D , from a clique (a Complete graph) on four vertices.and take two additional vertices , denoted by E and F.Connect*u*to one vertex on$C_{16n}$and*v*to another vertex on Letthe edge connecting*u*and*v*be denoted as *e*.

This graph*G*now has16*n*+2vertices and contains the edge *e*. We can easily

see that$C16n$is a Hamiltonian cycle in*G*containing as *e*.

Now , let 's describe the steps:

1. Starting with C_{16n}Begin traversing C_{16n}until you reach edge
2. Instead of continuing around $C16n$, traverse edge $C16n$ *e* to reach vertex v
3. From *v*, traverse back to *u* through edge *e*.
4. Continue traversing $C_{16n}$from *u* until you reach the starting vertex of C_{16n}. This completes the first cycle, consisting of 16*n* steps

Lemma 3.3.

Assume that G is a graph, C is a Hamiltonian cycle in G, and C=P_1 , P_2 , ..., P_k Hamiltonian paths obtained by performing the algorithm with the initial edge e=[v_1 , v_2]. Assume, moreover, that H is a block in G such that vertices of H form an interval in C and no edge of H is incident with v_1 . Let 1=i_1 , ..., i_l be a list of i's such that the vertices of H form an interval in Pi ; let G',P'_{ij} be a graph or a path obtained from G or P_{ij} by collapsing H to a vertex. Then i_l=k and $P'i_1$, ..., $P'_{i1}$are Hamiltonian paths in G'obtained by performing the initial edge vertex incident with In . the graph G_2 . Keep for that edge the label In . Rename the edge I in H by I_{n+1} and vertex w by w_{n+1}

Lemma 3.4 :Hamiltonian cycle is odd and even is NP- complete .

Proof :

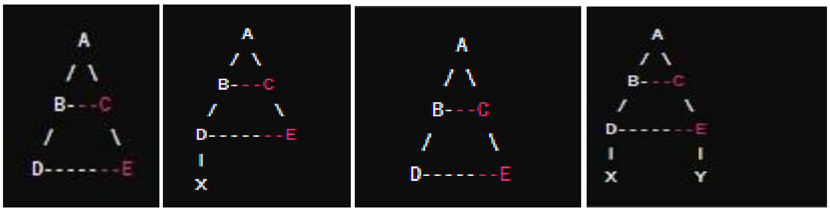
Let's start with an instance of the Hamilktonian cycle problem, represented by an arbitrary graph G.We'll construct a new graph $G'$by adding a single isolated vertex to G , making the total number of vertices odd.

1. A cycle that visits each vertex exactly once in a graph, is indeed NP-complete this true regardless of whether the number of vertices in the graph is odd or even
2. The complexity of determining whether a given graph has a Hamiltonian cycle specifically when it has an odd number of vertices or when it has an even number of vertices, it's important to note that both variants are Np- Complete.
3. Hamiltonian cycle each vertiex exactly once (Reduction)
4. Transforming to/ood even variant a polynomial -time reduction that transforms any instance of the Hamiltonian cycle problem into an instance of the odd or even variant.
5. If the orginal graph has a Hamiltonian cycle, then the transformed graph has a Hamiltonian cycle of the same parity.
6. The Hamiltonian cycle problem is Np-complete and that instances of its odd or even variants can be reduced to it in polynomial time, the odd or even variants are also NP- complete.

Algorithm :

```
Def Hamiltonian_ cycle (graph):
n= len(graph)
Path= [None]*n
Def is _ valid (vertex ,Pos):
If graph[path[pos-1][vertex]==0
Return False
If vertex in path[:pos]:
Return False
Return
Def Hamiltonian _util(pos):
If pos==n if graph[path[pos-1][path[0]]==1
Return True
For vertex in range (1,n):
If is_ valid (vertex,pos):
If Hamiltonian_util.pos+1):
Return True
If pos==n
If graph[path[pos-1 `]][path[0]]==1
Return true Else:
Return False For vertex in range (1,n):
If is _valid (vertex,pos):
Path[pos]=vertex
If hamiltonian _util(pos+1)
Return True
Path[pos]=None
Return False
Path[0]=0
If not hamiltonain _util(1):
```

```
Prinit("No Hamiltonian Cycle exists" )
Return False
Prinit ("Hamiltonianb Cycle exists:")
Prinit(path)
Return True
# Examplke Usage
Graph =[
[0,1,0,1,0]
[1,0,1,1,1]
[0,1,0,0,1]
[1,1,0,0,1]
[0,1,1,1,0]
Hamiltonian _cycle(graphs)
```



1. original graph G 1.Transformed GraphG'(odd)
2. original graph G 1.Transformed GraphG'(Even)

Theorem 3.5 for any fixed $k \geq 3$, Hamiltonian cycle is odd and even is NP- complete Proof: by induction

Creating a computer algorithm to directly solve the NP-completeness of a problem isn't feasible because NP-completeness refers to the difficulty of solving a problem with respect to polynomial-time algorithms. Instead, I'll outline the steps to prove NP-completeness, focusing on reduction from a known NP-complete problem.
Let's say we're reducing from the Hamiltonian Cycle problem:

1. **Input**: Given an undirected graph G.
2. **Output**: Determine whether has a Hamiltonian cycle of odd or even length.
3. **Step 1: Define the problem**: Precisely define the problem as described above.
4. **Step 2: Show membership in NP**: Show that verifying a proposed solution (a Hamiltonian cycle of odd or even length) can be done in polynomial time.

5. **Step 3: Construct the reduction**:
- For each vertex v in the input graph G, create $k-1$ copies of v, resulting in a new graph G'.
- For each edge u, $v$in G, add $k-1$ new edges between the corresponding copies of u and v in $'G'$.
- Add $k-2$ new vertices and connect them to all copies of the original vertices G'.

6. **Step 4: Prove correctness of the reduction**:

- If *G* has a Hamiltonian cycle, it visits each vertex exactly once. In *'G'*, this translates to visiting each copy of each vertex exactly once and using the newly added vertices in the specific order to ensure odd or even length.
- If *G'* has a Hamiltonian cycle according to our conditions, then *G* has a Hamiltonian cycle.

7. **Step 5: Verify the reduction**:

• The reduction can be done in polynomial time, as it involves creating a new graph with a polynomial number of vertices and edges.

8. **Step 6: Conclude NP-completeness**:

• Since Hamiltonian Cycle is NP-complete and we've successfully reduced it to our problem in polynomial time, our problem is also NP-complete.

This outline provides a high-level algorithmic approach to proving NP- completeness. Implementing the algorithm involves coding the reduction process, which should be done carefully to ensure correctness and efficiency.

4.The HC graph

We will now study the special case of graph. The HC-k-graphs (hamiltonian cycle in a k- graph) is obviously polynomial for k = O, k = 1 and k = 2. We know from [2] that the HC- n- graphs is NP-complete. For any k 2 6, a k-regular graph cannot be planar (see [3]), then the problem is obviously polynomial. We will use the result of HC-n graphs- Complexity of the Hamiltonian cycle in regular graph problem How a Hamiltonian circuit passes through a vertex w.. A Hamiltonian circuit crosses two vertices w. planar problem to show that the HC4-regular-planar and the HC-S graphs are NP-complete.

Theorem 4.1.

The Hamiltonian cycles -4 is NP-complete.

To prove that Hc-4 is NP - complete, we need to show two things:

1. **HC (-4) is in NP** :a graph *G* and a proposed Hamiltonian cycle *C* in *G*, it's easy to verify in polynomial time whether *C* indeed visits every vertex exactly once and returns to the starting vertex. This verification involves checking that *C* is a simple cycle that contains every vertex of *G* One common NP-complete problem is the Hamiltonian Cycleproblem itself.

- For each vertex *vi* in *G*, we replace it with a gadget consisting of a cycle of four vertices, each connected to exactly two other vertices in the cycle.
- We connect the corresponding vertices in the gadgets according to the edges in *G*, ensuring that if there is an edge between vertices *vi* and *vj*in *G*, there is a path in *G'* connecting thecorresponding gadgets of *vi* and *vj*.

It's easy to see that the resulting graph *G'* is a -4 graph, and it has a Hamiltonian cycle if and only if *G* has a Hamiltonian cycle. This reduction can be done in polynomial time.

Therefore, since the standard Hamiltonian Cycle problem can be reduced to HC(-4) in polynomial time, and because the standard Hamiltonian Cycle problem is NP-complete, HC(-4) is also NP-complete.

Theorem 4.2.

The Hamiltonian cycles -5 is NP-complete.

Theorem 4.3: Every cubic graph G has an even number of Hamiltonian cycles, containing a given edge e. Prof:Every cubic graph G has an even number of Hamiltonian cycles containing a given edge, let's proceed with a formal argument. Let's denote the given edge as e. we will consider the Hamiltonian cycles that contain edge e and count them.

Let's choose edge e. Since e is part of the cubic graph G, it is adjacent to two other edges, let's denote them as e_1 and e_2. Without loss of generality, we can assume that e_1 and e_2 form a path with e, rather than a cycle, form a path with e, rather than a cycle, as considering them as a cycle would lead to the same Extend the path formed by e, e_1, and e_2 to form a Hamiltonian cycle. Since G is cubic, there is exactly one way to extend this path to form a Hamiltonian cycle Remain n-3 edges where n is the number of vertices in the graph. Since each vertex has degree 3, and we've already used one edge incident to each of the vertices on the path The graph obtained by deleting the edge e from G. This graph has two components, each connected to the endpoints of e. Let's denote these components as G_1 and G_2

The number of Hamiltonian cycles in a connected cubic graph with n vertices is known to be $\frac{(n-1)!}{2}$ Therefore, the number of Hamiltonian cycles in G_1 and G_2, denoted as $H(G_1)$and$H$ G_2) and respectively, are $(n1-1)!$ and Therefore, the total number of Hamiltonian cycles containing e is even. This holds for any cubic graph G and any edge e in G.

Total number of Hamiltonian cycles containing edge is

$H(G_1)$and G_2) and $H($ respectively, are $\frac{(n2-1)!}{2}$ and every cubic $\frac{(n2-1)!}{2}$

graph G has an even number of Hamiltonian cycles containing a given edge e, as each Hamiltonian cycle contributes an even number of covers to edge e

Theorem 4.3.

For any n≥1, there exists a graph G_n with 6n vertices, an edge e of G, and an initial Hamiltonian path C in G containing e, Let's construct the graph Gnstep by step.

a cycle graph C_6 with 6 vertices and 6 edges. Let's label the vertices as $v1,v2,v3,v4,v5,v6$ in cyclic order.For each vertex vi in C_6, add a triangle (3-vertex complete graph) $T_i$such that each vertex of $T_i$is connected to v_i.This step adds3 vertices and 3 edges for each vertex in C_6, resulting in a total of 6×3=186×3=18 vertices and 6×3=186×3=18 edges.For each pair of triangles $T_i$and T_{i+1} (where the indices are taken modulo 6), connect the corresponding vertices of $T_i$and T_{i+1} with an edge. This step adds 6 edges.

The resulting graph$G1$ has6+18=246+18=24verticesand6+18+6=306+18+6=30 edges. It also contains a Hamiltonian cycle that includesone of the added edges.

Now, let's generalize this construction for any n, For each additional n, repeatsteps 2 and 3. That is, for each vertex vi in the cycle graph $C6$, add a Ti, andconnect the corresponding vertices of adjacent triangles with an edge. This process adds $18n$ vertices and $24n$ edges.

Therefore, for any $n \geq 1$, the graph G_n has $6n$ vertices and $6n+6(n-1)=12n-6$ edges. Additionally, the initial Hamiltonian path C contains one of the edges added in step 3.

6nvertices

5.Conclusions

1. We have studied Hamiltonian cycles on graphs, by which we mean rectangular graphs. a complexity measure for Hamiltonian cycles on rectangular graphs,
2. Any Hamiltonian cycle can be transformed to any other Hamiltonian cycle using only a linear number of transposes, there by initiating a study of k- Hamiltonian cycles in graphs as we next describe.
3. Hamiltonian cycle graph $G_{m,n}$ to be the graph whose vertices are Hamiltonian cycles on an $m \times n$ graph G.
4. Two vertices u,v of $G_{m,n}$ have an edge between them if one can obtain u from v and vice versa by applying a single transpose operation.
5. The subgraph $G_{m,n,k}$ of $G_{m,n}$ contains exactly the Hamiltonian cycles with bend complexity k.
6. Our result shows that $G_{m,n,1}$ is a connected graph and that the diameter of $G_{m,n,1}$ is at most O(mn). We pose the question whether $G_{m,n,k}$ is a connected graph, where $k > 1$.

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