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Kernel Dynamics in Three Species Ecological Model with Distributed time delay

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Abstract:

To understand the influence of the delay kernel weight on the dynamics of the three species model with prey, predator and competitor. Here competitor is competing with both the prey and predator species. A distributed time lag is induced in prey predator interaction. The Mathematical model is characterized by the system of integro differential equations. We proved that the solutions are positive and bounded in R_3^+. The equilibrium point for co-existing state is identified. The local stability analysis is address at this point and shown that the system is asymptotically stable. Global dynamics is addressed by the constructing suitable Lyapunov's function. The influence of kernel weight is studied using numerical simulation. The exponential kernel is taken for investigation. The kernel weights are significant in system dynamics. We identify the range of kernel strengths which exhibits the Hopf bifurcation nature. The parameters are identified using numerical simulation where the system admits change over behaviour from stable to unstable nature.

Keywords: Prey, Predator, Competitor, Co-existing state, stability analysis, Hopf bifurcation

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Introduction

The elegance of nature is explained using mathematical methods are prominent in the recent era. Mathematicians attracted to study the ecological and biological phenomenon. Differential equations are pioneering in this field. Keeping Natural resources in harmony entire ecological phenomena in balance. The study of ecological balance in terms of stability analysis using differential equations. The basic population growth models are well posed and initiated by Lokta [1] and Volterra [2].The ecological and epidemic models using mathematical techniques. Models explore real world problems are of non-linear in nature. The solutions of these models are quite difficult or sometimes impossible. To elucidate this, qualitative approach gained a significant role to study the stability aspects of ecological models. provide adequate mathematical approach to investigate the qualitative analysis of complex population models.

Prey-predator interactions are universal biological relationship in ecology. This relation attracted attention of many biologists, mathematicians. The prey predator dynamics with fractional order differential equations and Hopf bifurcation analytics are well versed by [3]. The memory effect in predator and fear effect in prey species and its global dynamics was dealt by [4]. The harvesting of ecological resources is extensively studied during this era. Recently [5] explored the dynamics of bio-economic model with Michalis mention type harvesting in prey species and conclude that over harvesting leads to the bifurcation. Chen [6] studied the Hopf bifurcation analysis of diffusive prey-predator system. Wie Liu [7] explore the direction of Hopf bifurcation in the Gause type prey-predator model with Michaelis-mention type harvesting model. Ranjith [8] discussed the crowding effects and its mechanics in prey-predator model with inter specific competition rates. Recently paparao [9,10] explored the harvesting efforts with constant effort and Michaelis-mention in two compartment prey-predator model and shown that the system is globally asymptotically stable.

In any ecological and biological phenomenon time delays are inevitable. Time delays mainly classified as discrete, continuous, and distributed type. The time delay term can be introduced in prey, predator and into the interaction of prey and predator terms. The time lags in the three species models exhibits rich dynamics. The distributed time lags in the three species models in different interaction terms are widely studied by paparao [11-14,16]. The stability aspects of system elaborated with distributed time lags and studied the local and global dynamics at equilibrium points. The system is asymptotically stable at co-existing state and the results are driven by numerical simulation with different delay kernel strengths. The time delay models with a prey, predator and ammesal model was dealt by [15] and observe the Hopf bifurcation tendency. The critical values of the model are identified using time lag as bifurcation parameter.

The Hopf bifurcation tendency in distributed type delay models are quite intersecting. So far this type of analysis was not considered. Despite these models, we proposed a three species model distributed time delay model with a prey, predator, and competitor. The dynamics of the model and instability criteria was disused using Hopf bifurcation analysis taking exponential kernel as weight function and parameter 'λ'taken as bifurcation parameter.

The paper is organized as follows. In section 2 we frame the model, section 3 equilibrium point, in section 4 local stability analysis in section 5 derived condition for existence of global stability analysis. Finally in section 6 numerical simulations is carried out in support of Hopf bifurcation analysis.

2. Formation of Mathematical model:

The proposed model with three species namely prey(N_1), predator (N_2) and competitor (N_3) with distributed time lag in the prey-predator interaction. Exponential growth model is proposed and is described by the following system of integro-differential equations.

$$\frac{dN_1}{dt} = a_1N_1 - \alpha_{11}N_1^2 - \alpha_{12}N_1\int_{-\infty}^{t} k_1(t-u)\, N_2(u)du - \alpha_{13}N_1N_3$$
$$\frac{dN_2}{dt} = a_2N_2 - \alpha_{22}{N_2}^2 + \alpha_{21}N_2\int_{-\infty}^{t} k_2(t-u)\, N_1(u)du - \alpha_{23}N_2N_3 \qquad (2.1)$$
$$\frac{dN_3}{dt} = a_3N_3 - \alpha_{33}N_3^2 - \alpha_{31}N_1N_3 - \alpha_{32}N_3N_2$$

Nomenclature:

parameter	Description
a_1	Growth rate of prey population
a_2	Growth rate of predator population
a_3	Growth rate of competitor population
N_1	Prey population
N_2	Predator population
N_3	Competitor population
α_{11}	Inter specific competition rate in prey species
α_{22}	Inter specific competition rate in predator species
α_{33}	Inter specific competition rate in competitor species
α_{12}	Prey predator interaction rate
α_{21}	Predator prey interaction rate

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α_{13}	Prey and competitor interaction rate
α_{31}	Competitor and prey interaction rate
α_{23}	Predator and competitor interaction rate
α_{32}	Competitor and predator interaction rate
$k_1(t-u)$ $k_2(t-u)$	Kernel strengths

Put t-u = z , in the equation (2.1) we get the following system of equations

$$\frac{dN_1}{dt} = a_1N_1 - \alpha_{11}N_1^2 - \alpha_{12}N_1\int_0^\infty k_1(z)\,N_2(t-z)dz - \alpha_{13}N_1N_3$$
$$\frac{dN_2}{dt} = a_2N_2 - \alpha_{22}{N_2}^2 + \alpha_{21}N_2\int_0^\infty k_2(z)\,N_1(t-z)dz - \alpha_{23}N_2N_3 \qquad (2.2)$$
$$\frac{dN_3}{dt} = a_3N_3 - \alpha_{33}N_3^2 - \alpha_{31}N_1N_3 - \alpha_{32}N_3N_2$$

Choose the kernels k_1 and k_2 such that

$$\int_0^\infty k_1(z)dz = 1, \int_0^\infty k_2(z)dz = 1\,, \int_0^\infty zk_1(z)dz < \infty, \& \int_0^\infty zk_2(z)dz < \infty \quad (2.3)$$

Assuming the solutions for the above model (2.2) as

$N_1 = A_1e^{\lambda t}, \qquad N_2 = A_2e^{\lambda t}, \qquad N_3 = A_3e^{\lambda t}$

The system equations (2.4) transform in to

$$N_1' = a_1N_1 - \alpha_{11}{N_1}^2 - \alpha_{12}N_1N_2k_1(\lambda) - \alpha_{13}N_1N_3$$
$$N_2' = a_2N_2 - \alpha_{22}{N_2}^2 + \alpha_{21}N_1N_2k_2(\lambda) - \alpha_{23}N_2N_3 \qquad (2.4)$$
$$N_3' = a_3N_3 - \alpha_{33}{N_3}^2 - \alpha_{31}N_1N_3 - \alpha_{32}N_2N_3$$

Here $k_1(\lambda)$is the Laplace transform of$k_1(z)$ and $k_2(\lambda)$ is the Laplace transform of$k_2(z)$

2.1 Existence of positive solutions and bounded property.

In this section we proved that the system (2.4) admits positive solutions and are also bounded.

Theorem 2.1: The system (2.4) possesses positive solutions in R_3^+

Proof: From the equation (2.4) we can write as follows

$$N_1(t) = N_{10}e^{\int_0^t (a_1 - \alpha_{11}N_1 - \alpha_{12}N_2k_1(\lambda) - \alpha_{13}N_3))dt} > 0$$

$$N_2(t) = N_{20}e^{\int_0^t (a_2 - \alpha_{22}N_2 + \alpha_{21}N_1k_2(\lambda) - \alpha_{23}N_3)dt} > 0$$

$$N_3(t) = N_{30}e^{\int_0^t (a_3 - \alpha_{33}N_3 - \alpha_{31}N_1 - \alpha_{32}N_2)dt} > 0$$

Hence all the solutions are positive inR_3^+

Theorem 2.2: The solutions of the system (2.4) are bounded.

Proof: A system of equations (2.4) can be written in the form

$$N_1' \leq \{a_1N_1 - \alpha_{11}N_1^2\}$$
$$N_2' \leq \{a_2N_2 - \alpha_{22}N_2^2\} \qquad (2.2.1)$$
$$N_3' \leq \{a_3N_3 - \alpha_{33}N_3^2\}$$

By solving the above equations, we can write as

$N_1(t) \leq \frac{a_1}{\alpha_{11}+ke^{-a_1t}}$, $N_2(t) \leq \frac{a_2}{\alpha_{22}+ke^{-a_2t}}$ and $N_1(t) \leq \frac{a_3}{\alpha_{33}+ke^{-a_3t}}$(2.2.2)

From the above equation (2.2.2), it is evident that the system (2.4) possesses bounded solutions.

Results and Discussion:

In this section we identify the co-existing state and analysing the stability both locally and globally at this point**.**

3. Equilibrium state

The co-existing state is obtained by equating $\frac{dN_i}{dt} = 0$, i=1,2,3 from the system (2.4),

E: Co-existing state:

$$\overline{N_1} = \frac{a_1(\alpha_{22}\alpha_{33} - \alpha_{23}\alpha_{32}) - \alpha_{12}k_1(\lambda)(a_2\alpha_{33} - a_3\alpha_{23}) + \alpha_{13}(a_2\alpha_{32} - a_3\alpha_{22})}{\alpha_{11}(\alpha_{22}\alpha_{33} - \alpha_{23}\alpha_{32}) + \alpha_{12}k_1(\lambda)(\alpha_{21}k_2(\lambda)\alpha_{33} + \alpha_{31}\alpha_{23}) - \alpha_{13}(\alpha_{21}k_2(\lambda)\alpha_{32} + \alpha_{31}\alpha_{22})}$$

$$\overline{N_2} = \frac{(a_2\alpha_{11}\alpha_{33} + a_1\alpha_{21}k_2(\lambda)\alpha_{33} + a_1\alpha_{31}\alpha_{23}) - (a_3\alpha_{11}\alpha_{23} + a_3\alpha_{21}k_2(\lambda)\alpha_{13} + a_2\alpha_{13}\alpha_{31})}{\alpha_{11}(\alpha_{22}\alpha_{33} - \alpha_{23}\alpha_{32}) + \alpha_{12}k_1(\lambda)(\alpha_{21}k_2(\lambda)\alpha_{33} + \alpha_{31}\alpha_{23}) - \alpha_{13}(\alpha_{21}k_2(\lambda)\alpha_{32} + \alpha_{31}\alpha_{22})}$$

$$\overline{N_3} = \frac{(a_3\alpha_{11}\alpha_{22} + a_3\alpha_{21}k_2(\lambda)\alpha_{12}k_1(\lambda) + a_2\alpha_{12}k_1(\lambda)\alpha_{31}) - (a_2\alpha_{11}\alpha_{32} + a_1\alpha_{21}k_2(\lambda)\alpha_{32} + a_1\alpha_{31}\alpha_{22})}{\alpha_{11}(\alpha_{22}\alpha_{33} - \alpha_{23}\alpha_{32}) + \alpha_{12}k_1(\lambda)(\alpha_{21}k_2(\lambda)\alpha_{33} + \alpha_{31}\alpha_{23}) - \alpha_{13}(\alpha_{21}k_2(\lambda)\alpha_{32} + \alpha_{31}\alpha_{22})}$$

. (3.1)

This equilibrium state exists only when, $\overline{N}_1 > 0, \overline{N}_2 > 0, \overline{N}_3 > 0$ (3.2)

4. Linear Stability analysis at the co-existing state:

Theorem: 4.The co-existing state $E\left(\overline{N_1}, \overline{N_2}, \overline{N_3}\right)$ is locally asymptotically stable if

$\alpha_{11}\alpha_{33} - \alpha_{13}\alpha_{31} > 0$

Proof: The variational matrix is given by

$$J = \begin{bmatrix} -\alpha_{11}\overline{N_1} & -\alpha_{12}\overline{N_1}k_1(\lambda) & -\alpha_{13}\overline{N_1} \\ \alpha_{21}\overline{N_2}k_2(\lambda) & -\alpha_{22}\overline{N_2} & -\alpha_{23}\overline{N_2} \\ -\alpha_{31}\overline{N_3} & -\alpha_{32}\overline{N_3} & -\alpha_{33}\overline{N_3} \end{bmatrix} \qquad (4.1)$$

With the characteristic equation $\lambda^3 + b_1\lambda^2 + b_2\lambda + b_3 = 0$ (4.2)

Here

$$b_1 = \left(\alpha_{11}\overline{N_1} + \alpha_{22}\overline{N_2} + \alpha_{33}\overline{N_3}\right)$$

$$b_2 = \left(\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21}k_1(\lambda)k_2(\lambda)\right)\overline{N_1N_2} + (\alpha_{11}\alpha_{33} - \alpha_{13}\alpha_{31})\overline{N_1N_3} + (\alpha_{22}\alpha_{33} - \alpha_{23}\alpha_{32})\overline{N_2N_3}$$

$$b_3 = \overline{N_1N_2N_3}(\alpha_{11}\alpha_{22}\alpha_{33} + \alpha_{12}\alpha_{21}\alpha_{33}k_1(\lambda)k_2(\lambda) - \alpha_{13}\alpha_{22}\alpha_{31} - \alpha_{11}\alpha_{23}\alpha_{32} + \alpha_{12}\alpha_{23}\alpha_{31}k_1(\lambda) + \alpha_{13}\alpha_{21}\alpha_{32}k_2(\lambda))$$

(4.3)

By Routh-Hurwitz criteria, the system is stable if $b_1 > 0$, $(b_1b_2 - b_3) > 0$ and $b_3(b_1b_2 - b_3) > 0$.

Clearly $b_1 = \left(\alpha_{11}\overline{N_1} + \alpha_{22}\overline{N_2} + \alpha_{33}\overline{N_3}\right) > 0$

By algebraic calculations

$$(b_1b_2 - b_3) = (\alpha_{11}{}^2\alpha_{22} + \alpha_{11}\alpha_{12}\alpha_{21}k_1(\lambda)k_2(\lambda))\overline{N_1}^2\overline{N_2} + (\alpha_{11}{}^2\alpha_{33} - \alpha_{11}\alpha_{13}\alpha_{31})\overline{N_1}^2\overline{N_3} + (\alpha_{22}{}^2\alpha_{33} + \alpha_{22}\alpha_{23}\alpha_{32})\overline{N_2}^2\overline{N_3} + (\alpha_{22}{}^2\alpha_{11} + \alpha_{22}\alpha_{12}\alpha_{21}k_1(\lambda)k_2(\lambda))\overline{N_2}^2\overline{N_1} + (\alpha_{11}\alpha_{33}{}^2 - \alpha_{33}\alpha_{13}\alpha_{31})\overline{N_3}^2\overline{N_1} + (\alpha_{22}\alpha_{33}{}^2 + \alpha_{33}\alpha_{23}\alpha_{32})\overline{N_3}^2\overline{N_2} + \overline{N_1}\,\overline{N_2}\,\overline{N_3}(2\alpha_{11}\alpha_{22}\alpha_{33} + \alpha_{12}\alpha_{23}\alpha_{31}k_1(\lambda) + \alpha_{13}\alpha_{21}\alpha_{32}k_2(\lambda))$$

$(b_1b_2 - b_3) > 0$ if $\alpha_{11}\alpha_{33} - \alpha_{13}\alpha_{31} > 0$ (4.4)

Also $b_3(b_1b_2 - b_3) > 0$ if $\alpha_{11}\alpha_{33} - \alpha_{13}\alpha_{31} > 0$

Hence, the co-existing state $E\left(\overline{N_1}, \overline{N_2}, \overline{N_3}\right)$ is locally asymptotically stable if.

$\alpha_{11}\alpha_{33} - \alpha_{13}\alpha_{31} > 0.$

5. Global Stability:

Theorem5.1: The co-existing state $E\left(\overline{N_1}, \overline{N_2}, \overline{N_3}\right)$ is globally asymptotically stable.

Proof: We considered the Lyapunov's function as

$$V\left(\overline{N_1}, \overline{N_2}, \overline{N_3}\right) = \sum_{i=1}^{3} N_i - \bar{N_\iota} - \bar{N_\iota}\log\left(\frac{N_i}{\bar{N_\iota}}\right) + \frac{1}{2}\alpha_{12}\int_0^\infty k_1(z)\int_{t-z}^{t}[N_2 - \overline{N_2}]^2\,du\,dz + \frac{1}{2}\alpha_{21}\int_0^\infty k_2(z)\int_{t-z}^{t}[N_1 - \overline{N_1}]^2\,du\,dz$$

(5.1)

The time derivate of 'V' along with the solutions of equations (2.1) is

$$V^1(t) = \sum_{i=1}^{3}\frac{[N_i - \overline{N_i}]}{N_i}N_i^1 + \frac{1}{2}\alpha_{12}\int_0^\infty k_1(z)[N_2 - \overline{N_2}]^2 dz - \frac{1}{2}\alpha_{12}\int_0^\infty k_1(z)[N_2(t-z) - \overline{N_2}]^2 dz + \frac{1}{2}\alpha_{21}\int_0^\infty k_2(z)[N_1 - \overline{N_1}]^2 dz - \frac{1}{2}\alpha_{21}\int_0^\infty k_2(z)[N_1(t-z) - \overline{N_1}]^2 dz$$

(5.2)

From the equation (5.2), we have

$$V'(t) = [N_1 - \overline{N_1}]\left(a_1 - \alpha_{11}N_1 - \alpha_{12}\int_0^\infty k_1(z)\, N_2(t-z)dz - \alpha_{13}N_3\right. \\ + [N_2 - \overline{N_2}]\left(a_2 - \alpha_{22}N_2 + \alpha_{21}\int_0^\infty k_2(z)\, N_1(t-z)dz - \alpha_{23}\right)$$

$$+ [N_3 - \overline{N_3}](a_3 - \alpha_{33}N_3 - \alpha_{31}N_1 - \alpha_{32}N_2) +$$

$$\tfrac{1}{2}\alpha_{12}[N_2 - \overline{N_2}]^2 + \tfrac{1}{2}\alpha_{21}[N_1 - \overline{N_1}]^2 - \tfrac{1}{2}\alpha_{12}\int_0^\infty k_1(z)[N_2(t-z) - \overline{N_2}]^2 dz -$$

$$\tfrac{1}{2}\alpha_{21}\int_0^\infty k_2(z)[N_1(t-z) - \overline{N_1}]^2 dz$$

By proper choice of $a_1, a_2 \& a_3$ and using the inequality $ab \le \frac{a^2+b^2}{2}$

$$\int_0^\infty k_1(z)[N_2(t-z) - \overline{N_2}]^2 \mathrm{dz} \le \int_0^\infty k_1(z)dz = 1$$

$$\& \int_0^\infty k_2(z)[N_1(t-z) - \overline{N_1}]^2 dz \le \int_0^\infty k_2(z)dz = 1,$$

$$= -\alpha_{11}(N_1 - \overline{N_1})^2 - \alpha_{22}(N_2 - \overline{N_2})^2 - \alpha_{33}(N_3 - \overline{N_3})^2 - \frac{(\alpha_{13} + \alpha_{31})}{2}[(N_1 - \overline{N_1})^2 + (N_3 - \overline{N_3})^2] \\ + \frac{1}{2}\alpha_{21}[N_1 - \overline{N_1}]^2 - \frac{1}{2}\alpha_{12}[N_2 - \overline{N_2}]^2 + \frac{(\alpha_{23} + \alpha_{32})}{2}[(N_2 - \overline{N_2})^2 + (N_3 - \overline{N_3})^2] - \frac{1}{2}(\alpha_{12} + \alpha_{21})$$

$$\le -\left|\left(\alpha_{11} + \frac{1}{2}\alpha_{31} + \frac{1}{2}\alpha_{21} - \frac{1}{2}\alpha_{12}\right)\right|(N_1 - \overline{N_1})^2 - \left|\left(\alpha_{22} - \frac{1}{2}\alpha_{12} + \frac{1}{2}\alpha_{21} - \frac{1}{2}\alpha_{32} - \frac{1}{2}\alpha_{23}\right)\right|(N_2 - \overline{N_2})^2 \\ - \left|\left(\alpha_{33} + \frac{1}{2}\alpha_{13} - \frac{1}{2}\alpha_{32} - \frac{1}{2}\alpha_{23}\right)\right|(N_3 - \overline{N_3})^2 - \frac{1}{2}|(\alpha_{12} + \alpha_{21})|$$

$$\Rightarrow V'(t) \le -\mu\sum_{i=1}^{3}[N_i - \overline{N_i}]^2 < 0 \text{ for}$$

$$\mu = \min\left(\alpha_{11} + \alpha_{22} + \alpha_{33} + \frac{1}{2}\alpha_{13} + \frac{1}{2}\alpha_{31} + \frac{1}{2}\alpha_{21} - \frac{1}{2}\alpha_{12} - \frac{1}{2}(\alpha_{23} + \alpha_{32})\right)$$

Therefore, the system is globally stable at the co-existing state $E_2\left(\overline{N_1}, \overline{N_2}, \overline{N_3}\right)$

6 Numerical Simulations:

Representation of graphs A: Shows the time series flow B. Phase portrait.

Example 6.1: Let a_1=1.5, a_2=2.65, a_3=3.45, α_{11}=0.1, α_{12}=0.5, α_{13}=0.01, α_{21}=0.5, α_{22}=0.2, α_{23}=0.4, α_{31}=0.2, α_{32}=0.2, α_{33}=0.2, N_1 =10, N_2 =15, N_3 =15.

The systems of equations (2.2) with exponential delay kernel weights

$k_1(z) = k_2(z) = ae^{-az} for\ a > 0$becomes.

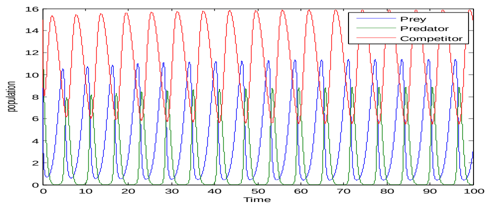
$$N_1' = a_1 N_1 - \alpha_{11} N_1^2 - \alpha_{12} N_1 N_2 \frac{\lambda}{\lambda + a} - \alpha_{13} N_1 N_3$$

$$N_2' = a_2 N_2 - \alpha_{22} N_2^2 + \alpha_{21} N_1 N_2 \frac{\lambda}{\lambda + a} - \alpha_{23} N_2 N_3 \qquad (6.1)$$

$$N_3' = a_3 N_3 - \alpha_{33} N_3^2 - \alpha_{31} N_1 N_3 - \alpha_{32} N_2 N_3$$

The system of equation (6.1) are simulated using the parametric values shown in example (6.1) with different kernel strengths $\lambda \& a$ are shown below.

Case (i) : for λ=0.005, *a*= 1



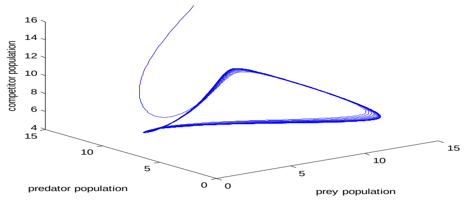
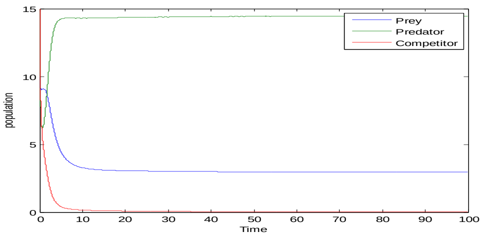


Figure 6.1(A) ***Figure 6.1(B)***

For the certain parametric values ofλ=0.005, ***a***= 1. The system exhibits unstable behaviour characterized by the unbounded oscillations in the three populations. However, when λ=0.005 fixed and the higher values of ***a (1 to 100)*** result in instability due to un bounded oscillations. For λ=0.005 and the lower values of ***a (***0.001) leads to stability and convergence to an equilibrium point E (3, 14, 0). The plots are shown below:

λ=0.005, *a*=0.001 E(3, 14, 0)



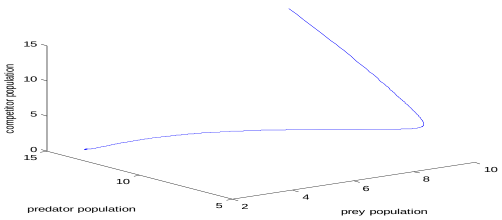
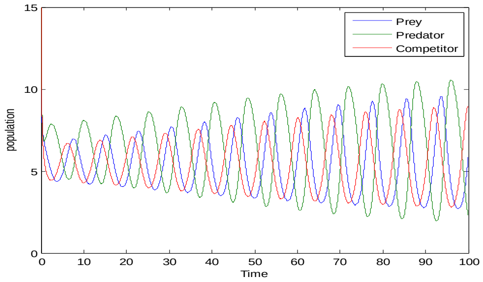


Figure 6.2(A) ***Figure 6.2(B)***

The stability range of the system is identified when λ=0.005 fixed range of parameter a value in[0.0001, 0.0015]. When 'a' exceed 0.0015 for fixed λ=0.005, the system transition from stable equilibrium to unstable equilibrium. For a > 0.0015 the system exhibits Hopf bifurcation for fixed λ=0.005.

Case (2) :λ=0.05, *a*=0.02



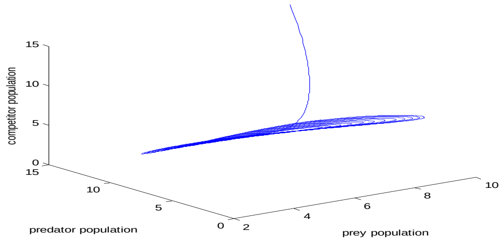
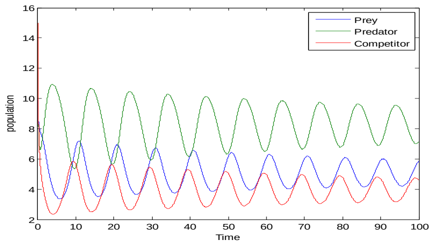


Figure 6.3(A) ***Figure 6.3(B)***

In stability nature is observed for λ=0.05and a= 0.02, the higher values of a ranging from the 0.02 to 100, results in in stability .For λ=0.05 and ***a***= 0.012 demonstrates the trajectories converging to words equilibrium point E (4, 11, 2). The evidence of time series plot and phase plane is given below.

λ=0.05, *a*=0.015



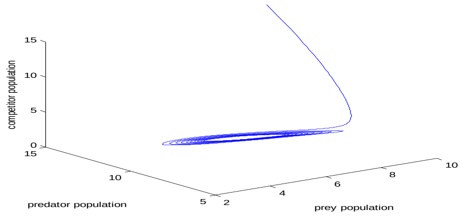
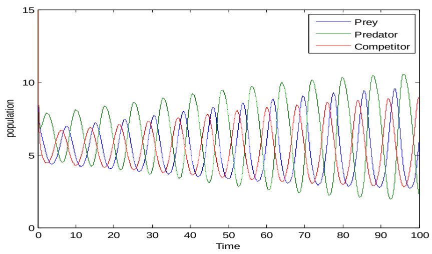


Figure 6.4(A) ***Figure 6.4(B)***

The stability range of parametric values 'a' from [0.0001, 0.015] when λ=0.05 fixed shown in the above graph. When 'a' exceeds shown 0.015 the system unstable. Hence for λ=0.05 and *a*> 0.015 indicating the qualitative change in system dynamics indicates bifurcation.

Case (3) :λ=0.5, *a*= 0.2



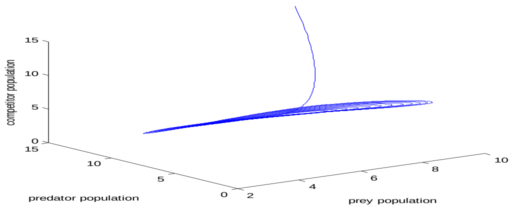
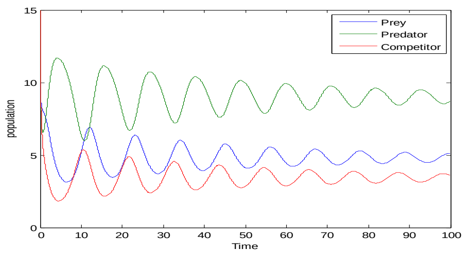


Figure 6.5(A) ***Figure 6.5(B)***

The time series and phase plane plots are evident that the system exhibits unstable nature for λ=0.5 and ***a***=0.2. However when λ=0.5 and a= 0.14 the system becomes asymptotically stable and converges to the equilibrium point E (5, 9, 4).

λ=0.5, *a*=0.14 E(5, 9, 4)



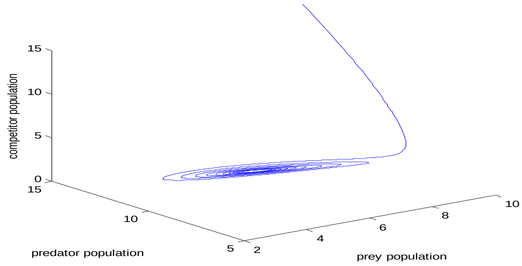


Figure 6.6 (A) ***Figure 6.6 (B)***

The system remains stable for λ=0.5 and a in the range [0.0001, 0.15]. When 'a' exceeds 0.15 for the fixedλ=0.5, the transition from stable equilibrium to unstable equilibrium evidence for Hopf bifurcation.

7. Conclusion

In the proposed model the following criteria are well illustrated.

The well posed ness of the mathematical model is discussed by studying the existence of positive and bounded solutions. The existences of positive solutions ensure that the dynamics described by the model are nonnegative and roundedness ensures the population should not grow indefinitely.

Local stability criteria are established and proved that the system is locally asymptotically stable at co-existing state if $\alpha_{11}\alpha_{33} - \alpha_{13}\alpha_{31} > 0$. A suitable Lyapunov's function is constructed to prove the global existence of the system. Numerical simulation is performed by varying the parameters specially the strengths of exponential delay kernel weights (λ&*a*). Different values of these kernel strengths (λ&*a*) lead to change in the system dynamics between stable and unstable equilibrium.

The critical values of (λ&*a*) are identified leading to a Hopf bifurcation. Hopf bifurcation characterizes the qualitative behavior of the system often involving unbounded oscillation. The switch over behavior is presented below via critical values of the model.

Table: 7.1: Critical values for (λ&*a)* for Hopf bifurcation Analysis

S.No.	Delay in the interaction of the species	Kernel Strengths		
		Values of λ considered	The range of ***a*** so that the system is stable	Hopf bifurcations, values (System becomes unstable)
1	Prey and the Predator	0.005	[0.0001, 0.0015]	λ = 0.005 &*a*> 0.0015

		0.05	[0.0001, 0.015]	λ = 0.05 &*a*> 0.015
		0.5	[0.0001, 0.15]	λ = 0.5 &*a*> 0.15

Therefore the delay arguments are significant to convert the stable equilibrium in to unstable equilibrium. So, the system undergoes Hopf bifurcation for some critical values of λ&*a* shown in the above table.

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