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The Total Certified Domination Number for Indu-Bala Product of Some Graphs

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Abstract

The Total Certified Domination Number aims for a balance between the concepts Certified domination and Total Domination providing better coverage with fewer dominating vertices. A total dominating et D is said to be total certified dominating set if every vertex in D has zero or at least two neighbours in $V(G) - D$.The minimumcardinality of a total certified dominating set is called the Total Certified Number and is denoted by$\gamma_{tcer}(G)$. This total certified domination concept provides a different approach to calculate the number of vertices required to dominate the paths and cycles based on the residue of n_1 modulo 4.For the two graphsG_1 andG_2,the Indu-Bala product$G_1 \blacktriangledown G_2$ be given from two disjoint copies of the join$G_1 \vee G_2$where the corresponding vertices of two copies of$G_2$are connected by an edge.In this article, the total certified domination concept is applied in Indu-Bala Product for some standard graphs and some important results and theorems are derived based on the findings.

Keywords: Domination Number, Total Domination Number, Certified Domination Number, Total Certified Domination Number, Indu-Bala Product.

AMSSubjectClassification:05C69

1. Introduction

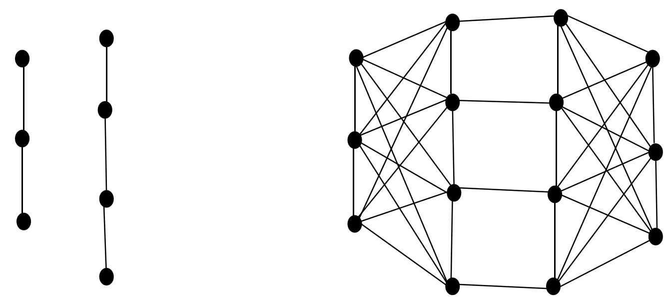
Let $G = (V, E)$ be a simple, undirected finite graph without multiple edges and loops. The graph $G = (V, E)$contains$n = |V|$ vertices and $m = |E|$ edges, unless it is indicated. Two vertices u and v are said to be adjacent if uv is an edge of G. The open neighbourhood of a vertex v in a graph G is defined as the set $N_G[v] = \{u \in V(G): uv \in E(G)\}$, while the closed neighbourhood of v in G is defined $N_G[v] = N_G(v) \cup \{v\}$. For any vertex v in a graph G, the number of vertices adjacent to v is called the degree of v in G, denoted by $deg_G(v)$. If the degree of a vertex is 0, it is called an isolated vertex, while if the degree is 1, it is called as an end-vertex. The minimum degree of vertices in G is defined by $\delta(G) =$

$\min\{\deg(v)/v \in V(G)\}$. The maximum degree ofvertices in G is defined by $\Delta(G) = \max\{\deg(v)/\ v \in V(G)\}$. A vertex v is called a universal vertex if $deg_G(v) = n - 1$.Domination in graph theory is anevolving field with significant advancements in theory and applications. A subset D of$V$ of a nontrivial graph G is called a dominating set of G if every vertex in $V - D$ is adjacent to at least one vertex in D.The domination number$\gamma(G)$ of G is the minimum cardinality taken over all the dominating sets in G.A dominating set D is said to be a Total Dominating Set if $\langle D \rangle$ has no isolated vertices. The minimum cardinality taken over all the total dominating sets is called the Total Domination Number and is denoted by $\gamma_t(G)$[1],[2].

The study of domination in graphs has rich history and extensive applications in various fields. Recent researches offer various extensions and variations of the concept, domination with unique properties and applications. The parameter Certified Domination Number was introduced in 2018 and defined asa dominating set$D$ of a graph G is said to be a Certified Dominating Set if every vertex in D has either zero or at least two neighbours in $V(G) - D$ [3],[4].The extension of the concept Total Certified Domination was introduced in 2019 and defined as atotal dominating set $D$of a graph$G = (V(G, E(G))$issaid to be a Total Certified Dominating Set if every vertices in D has either zero orat least two neighbours in $V(G) - D$[5]. The minimum cardinality of theTotal Certified Dominating set in$G$ is called theTotal Certified Domination Number of a graph$G$ and it is denoted by$\gamma_{tcer}(G)$.Total Certified Domination in graph theory that extends the idea of domination and gives a different perspective towards theoretical advancements in domination theory.

The Indu-Bala product of two graphs was introduced in 2016 and defined aslet G_1 and G_2 be two graphs of vertices $n_1$and n_2 and edges l_1 and l_2, respectively. The join operation $\vee$ of graphs G_1 and G_2 with disjoint vertex sets $V(G_1)$ and $V(G_2)$ and edge sets $E(G_1)$ and $E(G_2)$ is the union of the graphs, $G_1 \cup G_2$along with all the edges joining $V(G_1)$ and $V(G_2)$. For the two graphs G_1 and G_2, the Indu-Bala product $G_1 \blacktriangledown G_2$is given from the two disjoint copies of the join $G_1 \vee G_2$, where the corresponding vertices of two copies of G_2 are connected by an edge. The cardinality of vertices in$G_1 \blacktriangledown G_2$ is $|V(G_1 \blacktriangledown G_2)| = 2(n_1 + n_2)$ and edges is $|E(G_1 \blacktriangledown G_2)| = 2(l_1 + l_2 + n_1 n_2) + n_2$[6-10]. Total Certified Domination is applied in Indu-Bala graphs in order to analyse and verify the results and properties of the concept certified domination.

Examples 1.1. The two graphs $G_1 = P_3$, $G_2 = P_4$ and their Indu-Bala product $G = P_3 \blacktriangledown P_4$ is shown in the Figure 1.1



P_3 P_4 $P_3 \blacktriangledown P_4$

Figure 1.1

Definition1.1.ATotalDominatingSet$D$ofagraph $G = (V(G, E(G))$issaid to be the Total Certified Dominating Set if every vertices in D has either zero orat least two neighbors in $V(G) - D$. The minimum cardinality of the Total Certified Dominating Set in G is called the *Total Certified Domination Number* of a graph G anditisdenotedby$\gamma_{tcer}(G)$.

2. The Total Certified Domination Number ofIndu-Bala Product of Graphs

Theorem 2.1. Let the Indu-Bala product of two paths P_{n_1} and P_{n_2} be $G = P_{n_1} \blacktriangledown P_{n_2}$.

Then $\gamma_{tcer}(G) = \begin{cases} 4 & if \;\; n_1 = 2,3 \\ n_1 & if \;\; n_1 \equiv 0(mod4) \\ n_1 + 1 & if \; n_1 \equiv 1(mod4) \\ n_1 + 2 & if \; n_1 \equiv 2(mod4) \\ n_1 + 1 & if \; n_1 \equiv 3(mod4) \end{cases}$

Proof. Consider the Indu-Bala product of two paths P_{n_1} and P_{n_2} be $P_{n_1} \blacktriangledown P_{n_2}$. Let $\{v_1, v_2, \dots, v_{n_1}\}\{v_1', v_2', \dots, v_{n_1}'\}$ be the set of vertices of two copies of P_{n_1} and $\{u_1, u_2, \dots, u_{n_2}\}\{u_1', u_2', \dots, u_{n_2}'\}$ be the set of vertices of two copies of P_{n_2} for all $n_1 < n_2$. By the definition of the Indu-Bala product, all the vertices of two copies of P_{n_1} are adjacent with n_2 vertices of P_{n_2} and the corresponding vertices of two copies of P_{n_2} are connected by an edge.

Case(i) $n_1 = 2$ or 3. Let $D = \{v_1, v_2, v_1', v_2'\}$. Then D is the unique minimum γ_{tcer}-set of G so that $\gamma_{tcer}(G) = 4$.

Case(ii) $n_1 \equiv 0(mod4), n_1 = 4k, k \geq 1$.

Let $D = \{v_2, v_3, \dots, v_{n_1-2}, v_{n_1-1}\} \cup \{v_2', v_3', \dots, v_{n_1-2}', v_{n_1-1}'\}$. Then D is a γ_{tcer}-set of G so that $\gamma_{tcer}(G) \leq n_1$. Here, it is to beclaimedthat $\gamma_{tcer}(G) = n_1$. Suppose that $\gamma_{tcer}(G) \leq n_1 - 1$. Therefore, there is a minimum γ_{tcer}-set D' of G such that $|D'| \leq n_1 - 1$. First assume that $D' \subset D$. Let x be a vertex such that $x \in D$ and $x \notin D'$. With out loss of generality, let us assume that $x = v_{n_1-1}$. Then v_{n_1-1} is not dominated by any element of D', which is a contradiction. Therefore $\gamma_{tcer}(G) = n_1$.

Case(iii) $n_1 \equiv 1(mod4), n_1 = 4k + 1, k \geq 1$.

Let $D = \{v_2, v_3, \dots, v_{n_1-3}, v_{n_1-2}, v_{n_1-1}\} \cup \{v_2', v_3', \dots, v_{n_1-3}', v_{n_1-2}', v_{n_1-1}'\}$. Then D is a γ_{tcer}-set of G so that $\gamma_{tcer}(G) \leq n_1 + 1$. Here, it is to be claimed that $\gamma_{tcer}(G) = n_1 + 1$. Suppose that $\gamma_{tcer}(G) \leq n_1$. Therefore, there is a minimum γ_{tcer}-set D' of G such that $|D'| \leq n_1$. First assume that $D' \subset D$. Let x be a vertex such that $x \in D$ and $x \notin D'$. With out loss of generality, let us assume that $x = v_{n_1-1}$. Then v_{n_1-1} is not dominated by any element of D', which is a contradiction. Therefore $\gamma_{tcer}(G) = n_1 + 1$.

Case(iv) $n_1 \equiv 2(mod4), n_1 = 4k + 2, k \geq 1$.

Let $D = \{v_2, v_3, \dots, v_{n_1-4}, v_{n_1-3}, v_{n_1-2}, v_{n_1-1}\} \cup \{v_2', v_3', \dots, v_{n_1-4}', v_{n_1-3}', v_{n_1-2}', v_{n_1-1}'\}$. Then D is a γ_{tcer}-set of G so that $\gamma_{tcer}(G) \leq n_1 + 2$. Here, it is to be claimed that $\gamma_{tcer}(G) = n_1 + 2$. Suppose that $\gamma_{tcer}(G) \leq n_1 + 1$. Therefore, there is a minimum γ_{tcer}-set D' of G such that $|D'| \leq n_1 + 1$. First assume that $D' \subset D$. Let x be a vertex such that $x \in D$ and $x \notin D'$. With out loss of generality, let us assume that $x = v_{n_1-1}$. Then v_{n_1-1} is not dominated by any element of D', which is a contradiction. Therefore $\gamma_{tcer}(G) = n_1 + 2$.

Case(v) $n_1 \equiv 3(mod4), n_1 = 4k + 3, k \geq 1$.

Let $D = \{v_1, v_2, \dots, v_{n_1-6}, v_{n_1-5}, v_{n_1-2}, v_{n_1-1}\} \cup \{v_2', v_3', \dots, v_{n_1-6}', v_{n_1-5}', v_{n_1-2}', v_{n_1-1}'\}$. Then D is a γ_{tcer}-set of G so that $\gamma_{tcer}(G) \leq n_1 + 1$. Here, it is to be claimed that $\gamma_{tcer}(G) = n_1 + 1$. Suppose that $\gamma_{tcer}(G) \leq n_1$. Therefore, there is a minimum γ_{tcer}-set D' of G such that $|D'| \leq n_1$. First assume that $D' \subset D$. Let x be a vertex such that $x \in D$ and $x \notin D'$. With out loss of generality, let us assume that $x = v_{n_1-1}$. Then v_{n_1-1} is not dominated by any element of D', which is a contradiction. Therefore $\gamma_{tcer}(G) = n_1 + 1$.

Theorem 2.2. Let the Indu-Bala product of two cycles C_{n_1} and C_{n_2} be $G = C_{n_1} \blacktriangledown C_{n_2}$.

Then $\gamma_{tcer}(G) = \begin{cases} 4 & if \;\; n_1 = 2,3 \\ n_1 & if \;\; n_1 \equiv 0(mod4) \\ n_1 + 1 & if \; n_1 \equiv 1(mod4) \\ n_1 + 2 & if \; n_1 \equiv 2(mod4) \\ n_1 + 1 & if \; n_1 \equiv 3(mod4) \end{cases}$

Proof. Consider the Indu-Bala product of two cyclesC_{n_1} and C_{n_2} be$C_{n_1} \blacktriangledown C_{n_2}$. Let $\{v_1, v_2, \dots, v_{n_1}\}\{v_1', v_2', \dots, v_{n_1}'\}$ be the set of vertices of two copies of C_{n_1} and $\{u_1, u_2, \dots, u_{n_2}\}\{u_1', u_2', \dots, u_{n_2}'\}$ be the set of vertices of two copies of C_{n_2} for all $n_1 < n_2$. By the definition of the Indu-Bala product, all the vertices of two copies of C_{n_1} are adjacent with n_2 vertices of C_{n_2} and the corresponding vertices of two copies of $C_{n_2}$are connected by an edge.

Case(i) $n_1 = 2$ or 3. Let $D = \{v_2, v_3, v_2', v_3'\}$. Then D is the unique minimum γ_{tcer}-set of G so that $\gamma_{tcer}(G) = 4$.

Case(ii) $n_1 \equiv 0(mod4), n_1 = 4k, k \geq 1$.

Let $D = \{v_2, v_3, \dots, v_{n_1-2}, v_{n_1-1}\} \cup \{v_2', v_3', \dots, v_{n_1-2}', v_{n_1-1}'\}$. Then D is a γ_{tcer}-set of G so that $\gamma_{tcer}(G) \leq n_1$.Established that$\gamma_{tcer}(G) = n_1$.Assume that $\gamma_{tcer}(G) \leq n_1 - 1$. Therefore, there is a minimum γ_{tcer}-set D' of G such that $|D'| \leq n_1 - 1$. First assume that $D' \subset D$. Let x be a vertex such that $x \in D$ and $x \notin D'$. With out loss of generality, let us assume that $x = v_{n_1-1}$. Then v_{n_1-1} is not dominated by any element of D', which is a contradiction. Therefore $\gamma_{tcer}(G) = n_1$.

Case(iii) $n_1 \equiv 1(mod4), n_1 = 4k+1, k \geq 1$.

Let $D = \{v_2, v_3, \dots, v_{n_1-3}, v_{n_1-2}, v_{n_1-1}\} \cup \{v_2', v_3', \dots, v_{n_1-3}', v_{n_1-2}', v_{n_1-1}'\}$. Then D is a γ_{tcer}-set of G so that $\gamma_{tcer}(G) \leq n_1 + 1$.Established that$\gamma_{tcer}(G) = n_1 + 1$.Assume that$\gamma_{tcer}(G) \leq n_1$. Therefore, there is a minimum γ_{tcer}-set D' of G such that $|D'| \leq n_1 - 1$. First assume that $D' \subset D$. Let x be a vertex such that $x \in D$ and $x \notin D'$. With out loss of generality, let us assume that $x = v_{n_1-1}$. Then v_{n_1-1} is not dominated by any element of D', which is a contradiction. Therefore $\gamma_{tcer}(G) = n_1 + 1$.

Case(iv) $n_1 \equiv 2(mod4), n_1 = 4k+2, k \geq 1$.

Let $D = \{v_2, v_3, \dots, v_{n_1-4}, v_{n_1-3}, v_{n_1-2}, v_{n_1-1}\} \cup \{v_2', v_3', \dots, v_{n_1-4}', v_{n_1-3}', v_{n_1-2}', v_{n_1-1}'\}$. Then D is a γ_{tcer}-set of G so that $\gamma_{tcer}(G) \leq n_1 + 2$.Established that $\gamma_{tcer}(G) = n_1 + 2$.Assume that$\gamma_{tcer}(G) \leq n_1 + 1$. Therefore, there is a minimum γ_{tcer}-set D' of G such that $|D'| \leq n_1 + 1$. First assume that $D' \subset D$. Let x be a vertex such that $x \in D$ and $x \notin D'$. With out loss of generality, let us assume that $x = v_{n_1-1}$. Then v_{n_1-1} is not dominated by any element of D'. Here it contradicts our assumption. Hence $\gamma_{tcer}(G) = n_1 + 2$.

Case(v) $n_1 \equiv 3(mod4), n_1 = 4k+3, k \geq 1$.

Let $D = \{v_1, v_2, \dots, v_{n_1-6}, v_{n_1-5}, v_{n_1-2}, v_{n_1-1}\} \cup \{v_2', v_3', \dots, v_{n_1-6}', v_{n_1-5}', v_{n_1-2}', v_{n_1-1}'\}$. Then D is a γ_{tcer}-set of G so that $\gamma_{tcer}(G) \leq n_1 + 1$.Established that$\gamma_{tcer}(G) = n_1 + 1$. Assume that $\gamma_{tcer}(G) \leq n_1$. Therefore, there is a minimum γ_{tcer}-setD' of G such that $|D'| \leq n_1$. First assume that $D' \subset D$. Let x be a vertex such that $x \in D$ and $x \notin D'$. With out loss of generality, let us assume that $x = v_{n_1-1}$. Then v_{n_1-1} is not dominated by any element of D', which is a contradiction so that $\gamma_{tcer}(G) = n_1 + 1$.

Theorem 2.3. Let the Indu-Bala product of two graphs P_{n_1} and C_{n_2} be $G = P_{n_1} \blacktriangledown C_{n_2}$.

$$\text{Then } \gamma_{tcer}(G) = \begin{cases} 4 & if \ \ n_1 = 2,3 \\ n_1 & if \ \ n_1 \equiv 0(mod4) \\ n_1 + 1 & if \ n_1 \equiv 1(mod4) \\ n_1 + 2 & if \ n_1 \equiv 2(mod4) \\ n_1 + 1 & if \ n_1 \equiv 3(mod4) \end{cases}$$

Proof. Consider the Indu-Bala product of two graphsP_{n_1} and C_{n_2} be$P_{n_1} \blacktriangledown C_{n_2}$. Let $\{v_1, v_2, \dots, v_{n_1}\}\{v_1', v_2', \dots, v_{n_1}'\}$ be the set of vertices of two copies of C_{n_1} and $\{u_1, u_2, \dots, u_{n_2}\}\{u_1', u_2', \dots, u_{n_2}'\}$ be the set of vertices of two copies of C_{n_2} for all $n_1 < n_2$. By the definition of the Indu-Bala product, all the vertices of two copies of C_{n_1} are adjacent with n_2 vertices of C_{n_2} and the corresponding vertices of two copies of C_{n_2} are connected by an edge.

Case(i) $n_1 = 2$ or 3. Let $D = \{v_1, v_2, v_1', v_2'\}$. Then D is the unique minimum γ_{tcer}-set of G so that $\gamma_{tcer}(G) = 4$.

Case(ii) $n_1 \equiv 0(mod4), n_1 = 4k, k \geq 1$.

Let $D = \{v_2, v_3, \ldots, v_{n_1-2}, v_{n_1-1}\} \cup \{v_2', v_3', \ldots, v_{n_1-2}', v_{n_1-1}'\}$. Then D is a γ_{tcer}-set of G so that $\gamma_{tcer}(G) \leq n_1$. It is to be claimed $\gamma_{tcer}(G) = n_1$. Suppose that $\gamma_{tcer}(G) \leq n_1 - 1$. Therefore, there is a minimum γ_{tcer}-set D' of G such that $|D'| \leq n_1 - 1$. First assume that $D' \subset D$. Let x be a vertex such that $x \in D$ and $x \notin D'$. With out loss of generality, let us assume that $x = v_{n_1-1}$. Then v_{n_1-1} is not dominated by any element of D', which is a contradiction. Therefore $\gamma_{tcer}(G) = n_1$.

Case(iii) $n_1 \equiv 1(mod4), n_1 = 4k + 1, k \geq 1$.

Let $D = \{v_2, v_3, \ldots, v_{n_1-3}, v_{n_1-2}, v_{n_1-1}\} \cup \{v_2', v_3', \ldots, v_{n_1-3}', v_{n_1-2}', v_{n_1-1}'\}$. Then D is a γ_{tcer}-set of G so that $\gamma_{tcer}(G) \leq n_1 + 1$. It is to be claimed $\gamma_{tcer}(G) = n_1 + 1$. Suppose that $\gamma_{tcer}(G) \leq n_1$. Therefore, there is a minimum γ_{tcer}-set D' of G such that $|D'| \leq n_1$. First assume that $D' \subset D$. Let x be a vertex such that $x \in D$ and $x \notin D'$. With out loss of generality, let us assume that $x = v_{n_1-1}$. Then v_{n_1-1} is not dominated by any element of D', which is a contradiction. Therefore $\gamma_{tcer}(G) = n_1 + 1$.

Case(iv) $n_1 \equiv 2(mod4), n_1 = 4k + 2, k \geq 1$.

Let $D = \{v_2, v_3, \ldots, v_{n_1-4}, v_{n_1-3}, v_{n_1-2}, v_{n_1-1}\} \cup \{v_2', v_3', \ldots, v_{n_1-4}', v_{n_1-3}', v_{n_1-2}', v_{n_1-1}'\}$. Then D is a γ_{tcer}-set of G so that $\gamma_{tcer}(G) \leq n_1 + 2$. It is to be claimed $\gamma_{tcer}(G) = n_1 + 2$. Suppose that $\gamma_{tcer}(G) \leq n_1 + 1$. Therefore, there is a minimum γ_{tcer}-set D' of G such that $|D'| \leq n_1 + 1$. First assume that $D' \subset D$. Let x be a vertex such that $x \in D$ and $x \notin D'$. With out loss of generality, let us assume that $x = v_{n_1-1}$. Then v_{n_1-1} is not dominated by any element of D', which is a contradiction. Therefore $\gamma_{tcer}(G) = n_1 + 2$.

Case(v) $n_1 \equiv 3(mod4), n_1 = 4k + 3, k \geq 1$.

Let $D = \{v_1, v_2, \ldots, v_{n_1-6}, v_{n_1-5}, v_{n_1-2}, v_{n_1-1}\} \cup \{v_2', v_3', \ldots, v_{n_1-6}', v_{n_1-5}', v_{n_1-2}', v_{n_1-1}'\}$. Then D is a γ_{tcer}-set of G so that $\gamma_{tcer}(G) \leq n_1 + 1$. It is to be claimed $\gamma_{tcer}(G) = n_1 + 1$. Suppose that $\gamma_{tcer}(G) \leq n_1$. Therefore, there is a minimum γ_{tcer}-set D' of G such that $|D'| \leq n_1$. First assume that $D' \subset D$. Let x be a vertex such that $x \in D$ and $x \notin D'$. With out loss of generality, let us assume that $x = v_{n_1-1}$. Then v_{n_1-1} is not dominated by any element of D', which is a contradiction. Therefore $\gamma_{tcer}(G) = n_1 + 1$.

Theorem 2.4. Let the Indu-Bala product of two graphs K_{n_1} and P_{n_2} be $G = K_{n_1} \blacktriangledown P_{n_2}$.

Then $\gamma_{tcer}(G) = 4$.

Proof. Consider the Indu-Bala product of two graphs K_{n_1} and P_{n_2} be $K_{n_1} \blacktriangledown P_{n_2}$. Let $\{v_1, v_2, \ldots, v_{n_1}\}\{v_1', v_2', \ldots, v_{n_1}'\}$ be the set of vertices of two copies of C_{n_1} and $\{u_1, u_2, \ldots, u_{n_2}\}\{u_1', u_2', \ldots, u_{n_2}'\}$ be the set of vertices of two copies of P_{n_2}. By the definition of the Indu-Bala product, all the vertices of two copies of K_{n_1} are adjacent with n_2 vertices of P_{n_2} and the corresponding vertices of two copies of P_{n_2} are connected by an edge. Let $D = \{v_1\} \cup \{v_1'\}$. Then D is a certified dominating set of G so that $\gamma_{tcer}(G) = 2$. Since G contains isolated vertices D is not a total certified dominating set of G and so $\gamma_{tcer}(G) \geq 3$. It is easily verified that no three element subset of G

is not a total certified dominating set of G and $\gamma_{tcer}(G) \geq 4$. Consider the set $D = \{v_1, v_2\} \cup \{v_1', v_2'\}$. Then $D$is a total certified dominating set of G so that $\gamma_{tcer}(G) = 4$.

Theorem 2.5. Let the Indu-Bala product of two graphs C_{n_1} and S_{1,n_2-1} be $G = C_{n_1} \blacktriangledown S_{1,n_2-1}$, $n_1 \geq 3, n_2 \geq 2$. Then $\gamma_{tcer}(G) = 2$.

Proof. Consider the Indu-Bala product of two graphsC_{n_1} and S_{1,n_2-1} be$C_{n_1} \blacktriangledown S_{1,n_2-1}$. Let $\{v_1, v_2, \ldots, v_{n_1}\}\{v_1', v_2', \ldots, v_{n_1}'\}$ be the set of vertices of two copies of C_{n_1} and $\{u_1, u_2, \ldots, u_{n_2}\}\{u_1', u_2', \ldots, u_{n_2}'\}$ be the set of vertices of two copies of S_{1,n_2-1}. By the definition of the Indu-Bala product, all the vertices of two copies of C_{n_1} are adjacent with n_2 vertices of S_{1,n_2-1} and the corresponding vertices of two copies of $S_{1,n_2-1}$are connected by an edge.Let $D = \{u_1\} \cup \{u_1'\}$. Then D is a totalcertified dominating set of G so that $\gamma_{tcer}(G) = 2$.

Observations:

(i)Let the Indu-Bala product of two graphs P_{n_1} and S_{1,n_2-1} be$G = P_{n_1} \blacktriangledown S_{1,n_2-1}, n_1 \geq 2, n_2 \geq 2$. Then $\gamma_{tcer}(G) = 2$.

(ii)Let the Indu-Bala product of two graphs K_{n_1} and C_{n_2} be$G = K_{n_1} \blacktriangledown C_{n_2}, n_1 \geq 3, n_2 \geq 3$. Then $\gamma_{tcer}(G) = 4$.

(iii)Let the Indu-Bala product of two graphs W_{n_1} and P_{n_2} be$G = W_{n_1} \blacktriangledown P_{n_2}, n_1 \geq 3, n_2 \geq 3$. Then $\gamma_{tcer}(G) = 4$.

(iv)Let the Indu-Bala product of two graphs W_{n_1} and C_{n_2} be$G = W_{n_1} \blacktriangledown C_{n_2}, n_1 \geq 4, n_2 \geq 3$. Then $\gamma_{tcer}(G) = 4$.

(v)Let the Indu-Bala product of two graphs K_{n_1} and S_{1,n_2-1} be$G = K_{n_1} \blacktriangledown S_{1,n_2-1}, n_1 \geq 4, n_2 \geq 3$. Then $\gamma_{tcer}(G) = 4$.

(vi)Let the Indu-Bala product of two graphs K_{n_1} and W_{n_2} be$G = K_{n_1} \blacktriangledown W_{n_2}, n_1 \geq 3, n_2 \geq 4$. Then $\gamma_{tcer}(G) = 2$.

(vii)Let the Indu-Bala product of two graphs K_{n_1} and K_{n_2} be$G = K_{n_1} \blacktriangledown K_{n_2}, n_1, n_2 \geq 2$. Then $\gamma_{tcer}(G) = 2$.

(viii)Let the Indu-Bala product of two graphs W_{n_1} and S_{1,n_2-1} be$G = W_{n_1} \blacktriangledown S_{1,n_2-1}, n_1 \geq 4, n_2 \geq 2$. Then $\gamma_{tcer}(G) = 4$.

3.Conclusion

The study of the Total Certified Domination Number on the Indu-Bala product of graphs introduces a novel concept with significant theoretical and practical implications. It broadens the understanding of domination parameters in graph theory and opens up numerous avenues for future research, particularly in the areas of network design and optimization. The introduction of the New Parameter Total Certified Domination Number specifically focuses on the contribution towards the emerging trends in the field of graph theory and its application on Indu-Bala Product shows the future directions for further research in both the concepts.

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