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An Attempt to Estimate the Parameters of Janoschek Growth Model and Sloboda Growth Model

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ABSTRACT:

The primary emphasis of this paper is to study about two nonlinear growth models, the Janoschek growth model, and Sloboda growth model. The key aim is to introduce new methods of estimation for estimating the parameters in the Janoschek growth model and the Sloboda growth model. Examining this paper also offers more effective approaches to address the properties inherent in the models. While conventional nonlinear optimization techniques typically demand extensive computational efforts for the estimation of parameters. In contrast, the methods introduced herein, offer simplicity of understanding, making them suitable for smaller datasets, and enhancing their relevance across diverse contexts.

Keywords: Parameter Estimation, Nonlinear Growth Model, Janoschek Growth Model, Sloboda Growth Model

1. Introduction

The enduring fascination with predicting the future finds expression in mathematical modelling, a systematic approach to translating real-life dynamics into mathematical constructs employing mathematical language, ideas and concepts. Models, distilled representations of reality into mathematical abstractions, find wide-ranging applications across natural and engineering sciences including social sciences, statistics and game theory. Mathematical modelling significance extends to predicting population dynamics, disease propagation, optimizing decision-making precision, system control and the utilization of contemporary computational capabilities. Economically and socially, it contributes to rational planning and regulation. In essence, mathematical modelling emerges as an indispensable tool, unravelling the complexities of systems, providing strategic insights and effectively addressing challenges across diverse domains.

Janoschek model was initially introduced by Janoschek and subsequently modified by Gille and Salomon [11]. Keeping the properties unaltered without much change, a further modified iteration of this model was utilized during the study of COVID-19 outbreak in China [9] provided by,

$$y = a\left(1 - be^{-kt^c}\right) \tag{1}$$

Where y is the equivalent growth size at time t, a is the limiting value, this is referred to as the hypothetically estimated end value, b is the lower asymptotic, k is an uncharacterized parameter and c is the shape parameter that alters the degree of size at the point of inflection.

The Sloboda equation, introduced by German Biostatistician B Sloboda [4] has found diverse applications across life sciences, social sciences and physical sciences for representing growth phenomena. Gangying and Weitang [5] defined the Sloboda model, applying it successfully to Chinese fir plantations. Lee, demonstrated the superiority of the Sloboda model over six other sigmoidal growth curves when determining the most suitable model for Pinus Radiata based on age data [2]. Notably, Celic et al. [9] utilised the Sloboda model in their research on the spread of the COVID-19 virus outbreak in China. The Sloboda model is given by,

$$y = ae^{-be^{-kt^c}} \tag{2}$$

This paper aims to provide a brief discussion of two non linear growth models, namely, Jnaoschek growth model and Sloboda growth model, by examining and studing various properties attributed by these models. From prior discussions, it has been observed that these models find wide range of application in various contexts for representing growth phenomenon. This paper aims to introduce new methods of estimation for fitting the two candidate non linear growth models, most of which requires less computational efforts and can be applied to any growth data. This will offer a basic tool for researchers who have limited experience in the application of more advanced models. The newly introduced models are detailed in section 5 of this paper.

2. Literature Review

Originating in 1957, the Janoschek model underwent refinements during research on ducks by Gille and Salmon [11]. This led to the discovery of a distinct growth-length relationship in adult ducks. Subsequent applications included investigating rat body weight gain [14], uncovering maximal growth rates in male and female rats. Longitudinal biopsies [15] from White Pekin Ducks highlighted the superiority of FTG fibres over FTO in terms of diameter, emphasizing greater accuracy and relative growth rates. Further applications examined Pekins' growth [12], revealing faster development in specific muscles compared to mallards and Muscovites. A comprehensive study on various duck breeds demonstrated that organ growth outpaced weight gain, with the oesophagus showing similar developmental characteristics to body weight [13].

The Janoschek model's adaptability extended to bone growth in gulls [6], establishing a growth model for turbot fish [16], and proving the best fit for chukar partridge development data [1]. By 2019, applying the Janoschek growth curve model to observe the average values of various organs during incubation in JQ showcased its versatility across different species and components [3].

Biostatistician B. Sloboda introduced a new sigmoidal model [4], a generalized form of the Gompertz function, utilizing first-order differential equations. Widely applied across sectors for growth representation, Gangying and Weitang [5] validated the Sloboda equation on Chinese fir plantations, yielding satisfactory results. Later, Lee deemed the Sloboda model superior to six others in estimating Pinus radiata growth curves [2]. Tewari et al. employed Sloboda's algebraic difference equation, outperforming alternative models in analysing data from 22 sample plots across six districts in Gujarat, India [17]. In 2020, Celik. et al. utilized the Sloboda equation to model the COVID-19 outbreak in China, finding it apt for describing cumulative cases [9]. In the subsequent year, Suliman et al. [10] assessed top height and stand age data from 80 permanent plots, fitting them into eight generalized algebraic equations. Notably, the Sloboda equation demonstrated the best results in both biological credibility and model accuracy. This underscores its robust applicability across diverse domains.

3. Material and Methods

This paper focuses on the in-depth discussion of two non linear growth models, namely, Janoschek growth model and Sloboda growth model. The principal objective is to introduce new methods of estimation for determining the parameters of associated with these models. The proposed methods are intented to offers a more reliable approach for analysing nonlinear growth dynamics descried by Janoschek model and Sloboda model. The integral forms of the models are given in the following table.

Table 1: Integral forms of the candidate nonlinear growth models

Model name	Integral form of the model	Reference
Janoschek Model	$y = a\left(1 - be^{-kt^c}\right)$	[9]
Sloboda Model	$y = ae^{-be^{-kt^c}}$	[4]

Where y is the response variable, t is the independent variable and a, b, c and k are the parameters of the respective models.
To estimate the parameters, most of the literature reviewed in this paper employs some well-established algorithm. This paper aims to introduce some new methods of estimation to estimate the parameters of the respective models, facilitating that one can easily fit the models without using any dearly won software.
This paper also attempts to discuss different fundamental properties of the candidate models which may aid in comprehending the model thoroughly and provide initial parameter estimates essential during the application of iterative techniques.
The new methods of estimations and the properties of the prospective models are described in the following section.

4. Results and Discussion

The properties of the Janoschek growth model and Sloboda growth model are outlined below.

Properties of Janoschek growth model and Sloboda growth model
In this paper, an endeavour has been undertaken to discuss the different fundamental properties of the Janoschek growth model and the Sloboda growth model.
Fitting it by means of non-linear regression iteratively,

(a) The rate of growth with an increase in time is given by

$$y' = abckt^c e^{-kt^c} t^{c-1}$$

(b) The initial point at time $t = 0$ was calculated to give, $a - b$.

(c) If $p \leq 1$, the curve is simply exponential.

(d) For $p \geq 1$, a sigmoid pattern can be observed.

The Janoschek function can mainly be applied to describe the postnatal growth of individuals, but this function is also flexible with respect to its points of inflection, all of these characteristics that are desired in nonlinear growth models.

The Sloboda function is a Gompertz function with the addition of c as the power of t.

For an asymptote to exist, two possibilities arise, and depending on the sign of the parameter c, the asymptote can be generalized in two ways.

(a) If $c > 0$, it is required that $k > 0$, for the asymptote to be defined and then it has level a.

(b) If $c < 0$, the asymptote ae^{-b} and the function is increasing if $b > 0$ & $k < 0$ or if $b < 0$ & $k < 0$.

(c) The growth rate is given by,

$$y' = abckt^{c-1}e^{-be^{-kt^c}}e^{-kt^c}$$

(d) The inflection point of the curve can be approximated to be

$$\left(\frac{c-1}{ck(1-b)}\right)^{\frac{1}{c}}$$

By considering t reasonably large, hence e^{-kt^c} tends to 1.

Few more properties of the candidate mathematical models are delineated in the Table 2.

Table 2: Some mathematical properties of Janoschek growth model and Sloboda growth model

Properties/model	Janoschek growth model	Sloboda growth model
Integral form	$y = a\left(1 - be^{-kt^c}\right)$	$y = ae^{-be^{-kt^c}}$
Upper Asymptote	a	a
Starting point	$a - b$	ae^{-b}
Growth rate	$y' = abckt^c e^{-kt^c} t^{c-1}$	$y' = abckt^{c-1}e^{-be^{-kt^c}}e^{-kt^c}$
Relative Growth rate	$bckt^c t^{c-1}\left(1 - be^{-kt^c}\right)^{-1}$	$bckt^{c-1}e^{-kt^c}$
Second Derivative	$abcke^{-kt^c}t^c - 2\{(c-1) - t^c ck\}$	$abcke^{-be^{-kt^c}-kt^c}t^{c-2}[c - 1 + be^{-kt^{c(kct(c-1))}} - kct^c]$
Domain of the independent variable	$[0, \infty]$	$[0, \infty]$
Domain of the dependent variable	$[a - b, a]$	$[ae^{-b}, a]$

5. Methods of estimation

In the real world, predicting 100% outcomes for real life events based on basic assumptions and projections is exceedingly difficult. However, evaluating parameters using deeper techniques of estimations might assist eliminate assumption mistakes resulting in a better system output. So, a formalised procedure called estimation is used for estimating the value of one or more parameters, facilitating a strategic progression towards a more profound comprehension of the process. Other than the frequently used least squares methods, an array of alternative methods have been developed to estimate the parameters of mathematical models as per the requirements. To put this study together, several methods have been devised to estimate the parameters of the two nonlinear growth models namely, Janoschek growth model and Sloboda growth model.

Estimation of parameters of Jnaoschek growth model

Method 1

For this method of estimation, considering the parameter c known. Three observations from the observed data set are considered in the integral form of Janoschek growth model, such that,

$$y_i = a\left(1 - be^{-kt_i^c}\right);\ i = 1,2,3 \tag{3}$$

By iteratively employing division on the integral equations with point sets at $i = 1,2,3$ one can deduce the value of the variable b, which is,

$$b = \frac{y_2 - y_1}{y_2 be^{-kt_1^c} - y_1 be^{-kt_2^c}} \tag{4}$$

After finding b, the parameter k can be obtained by using any iterative method from the equation outlined below. Here, in this literature, Newton Raphson method is used. The equation is,

$$(Y_{23} - Y_{12})e^{-kt_2^c} + (Y_{12} - Y_{22})e^{-kt_3^c} + (Y_{22} - Y_{23})e^{-kt_1^c} = 0 \tag{5}$$

Through the estimation of the values assigned to parameter b and k, the parameter a can be obtained as,

$$a = \frac{y_1}{1 - be^{-kt_1^c}} \tag{6}$$

Reformulating equation (5), the estimated value of the parameter c is obtained, which is given by,

$$c = \frac{1}{\ln(t_1)} \ln\left(\frac{ab}{a - y_1}\right)^{\frac{1}{k}} \tag{7}$$

Method 2

For Janoschek growth model, in this method, we initially consider the parameters k and c are known. First, we attempt to detemine the parameters a and b. We divide the range of observations into three equal parts, that is, if we consider the total number of observations to be n, then we have to consider m such that $m = \left[\frac{n}{3}\right]$. Let S_1 is the sum of first m observations, S_2 is the sum of 2^{nd} m observations and S_3 is the sum of 3^{rd} m observations. After simplification, the model attains an equation of the form (8), which is given by,

$$(S_1 - S_2)R + (S_3 - S_1)Q + S_3P = 0 \tag{8}$$

where,

$P = e^{-kt_1^c} + e^{-kt_2^c} + \cdots + e^{-kt_m^c}$

$Q = e^{-kt_{m+1}^c} + e^{-kt_{m+2}^c} + \cdots + e^{-kt_{2m}^c}$

$R = e^{-kt_{2m+1}^c} + e^{-kt_{2m+2}^c} + \cdots + e^{-kt_{3m}^c}$

The equation (8) is with two unknowns a and b. Now, generalised Newton Raphson iteration techniques will be applied to determine the unknowns a and b. Once a and b are found, the subsequent steps involve applying a similar process to deduce the unknowns k and c. Divide the entire set of observations into two equal parts say m, therefore $m = \left[\frac{n}{2}\right]$. Subsequent numerical calculations yield's the value of c and k as

$$c = \frac{\sum_{i=m+1}^{2m} \ln Y - \sum_{i=1}^{m} \ln Y}{\sum_{i=m+1}^{2m} \ln i - \sum_{i=1}^{m} \ln i} \tag{9}$$

And

$$k = \frac{1}{m}\left(\sum_{i=1}^{m} \ln\ Y - c\sum_{i=1}^{m} \ln\ i\right) \quad (10)$$

where, $Y = \ln\frac{ab}{y-a}$

Method 3

Utilizing parameters derived from the aforementioned method 1 and assuming a and c are known, the integral form of Janoschek growth model may be reformulated as,

$$Y = 1 - be^{-kT} \quad (11)$$

where, $Y = \frac{y}{a}$, $T = t^c$

Now, we are considering any two arbitrary observations y_1 and y_2. The subsequent determination of the remaining two parameters k and b are as follows,

$$k = \frac{\log\left(\frac{1-Y_1}{1-Y_2}\right)}{T_2 - T_1} \quad (12)$$

And

$$b = (1 - Y_2)e_2^{kT} \quad (13)$$

Method 4

In this method, the values of a and c are taken as known by utilizing the method 2 and estimate the remaining two parameters k and b by using the same process used in Method 3.

Method 5

Proceeding with (11) from method 3 which is,

$$y = 1 - be^{-kT} \quad (14)$$

We employ the previously determined parameters a and c established in method 1. Upon reformulation of the provided equation, we obtain,

$$P = B - k \quad (15)$$

Which is a linear equation with unknown $B = \log b$ and k, where T is independent and $P = \log(1 - Y)$ is the dependent variable.

Now, the least squares method can be employed for the determination of parameters B and k. Following this, the parameter b can be derived as $b = e^B$. Through this process, the parameter values are obtained.

Method 6

In this approach, the parameters a and c are regarded as known, from method 2. The subsequent calculation process mirrors the steps outlined in method 5.

Estimation of parameters of Sloboda growth model

Method 1

In this method, we use three arbitrary points $t_i, i = 1,2,3$ from the given dataset. Consider n is total observation. Following a series of simplifications, we arrive at an equation of the form

$$(P + Q)e^{-kt_2^c} - Pe^{-kt_3^c} - Qe^{-kt_1^c} = 0 \quad (16)$$

Where, $P = \log\left(\frac{y_1}{y_2}\right)$ and, $Q = \log\left(\frac{y_2}{y_3}\right)$

Upon eliminating the parameters c and k, the remaining two parameters b and a can be assessed as follows

$$b = \left(\frac{\log\left(\frac{y_1}{y_2}\right)}{e^{-kt_2^c} - e^{-kt_1^c}}\right) \quad (17)$$

And

$$a = \frac{y_1}{e^{-be^{-kt_1^c}}} \quad (18)$$

Then, the Newton-Raphson method for two variables [16] is used to estimate the parameters c and k.

Method 2

For this method, the integral representation of the Sloboda growth model can be rewritten by considering logarithm on both sides as

$$Y = A - be^{-kt^c} \tag{19}$$

Where, $A = \log a$ and $Y = \log y$

Now consider three partial sums of the total observations, that is, consider $m = \left[\frac{n}{3}\right]$. Also, S_1 be the sum of first m Y_i, S_2 be the sum of second m Y_i and S_3 be the sum of third m Y_i. Then after few simplifications, we have equation of the form

$$(S_1 - S_2)R + (S_3 - S_1)Q + (S_2 - S_3)P = 0 \tag{20}$$

Where,

$P = e^{-kt_1^c} + e^{-kt_2^c} + \cdots + e^{-kt_m^c}$

$Q = e^{-kt_{m+1}^c} + e^{-kt_{m+2}^c} + \cdots + e^{-kt_{2m}^c}$

$R = e^{-kt_{2m+1}^c} + e^{-kt_{2m+2}^c} + \cdots + e^{-kt_{3m}^c}$

By employing the Newton Raphson method for two variables, one can determine the values of the parameters k and c can be obtained. Upon the determination of k and c, the parameters b and A can be estimated as

$$b = \frac{S_1 - S_2}{Q - P} \tag{21}$$

And

$$A = \frac{S_1 + bP}{m} \tag{22}$$

Then the following parameter a can be calculated as $a = e^A$.

Method 3

Under this method, the parameters a and c are considered as known from method 1. Then, the Sloboda equation will take the form of

$$Y = B - kT_i \tag{23}$$

Where, $Y = \log\left(\log\left(\frac{a}{y}\right)\right)$, $B = \log b$ and $T_i = t_i^c$

Utilizing the technique of least squares provide the requisite estimation for the parameters k and B can be obtained which is,

$$k = \left(\frac{\sum\left(Y_i - \frac{\sum Y}{n}\right)\left(T_i - \frac{\sum T_i}{n}\right)}{\sum\left(T_i - \frac{\sum T_i}{n}\right)}\right) \tag{24}$$

And

$$B = e^{\frac{k\sum T_i}{n} + \frac{\sum Y_i}{n}} \tag{25}$$

After having B, the parameter b can be acquired as $b = e^B$.

Method 4

This method closely follows the procedure outlined in method 3. In this method, the parameters c and a can be considered from method 2 instead of method 1.

Method 5

For this method, the parameters a and c are considered to be known from the method 1. We take two observations in the new reformulated integral form of the Sloboda's Growth model such that

$$Y_i = e^{-be^{-kT_i}} \quad (26)$$

where $i = 1,2$; $Y_i = \frac{y_i}{a}$ and $T_i = t_i^c$

The logarithmic transformations applied to both observations provide a pair of values for the parameter k. Setting these values equal to each other allows us to resolve the sought estimation of the parameter k and b which is given by,

$$k = \frac{\log b - \log P_1}{T_1} \quad (27)$$

And

$$b = e^{\frac{T_2 \log(P_1) - T_1 \log(P_2)}{T_2 - T_1}} \quad (28)$$

Subsequently, utilizing these estimations facilitates the derivation of the values for both a and c, ensuring a rigorous and precise characteristic of the underlying parameters.

Method 6

In this method, the parameters a and c are taken to be known and the method of equidistant points is employed to estimate the remaining parameters. The parameters a and c are derived from the method 2 and subsequently applied to estimate the other two unknown parameters, mirroring the process employed in method 5.

The models emphasised in this paper, as well as their characteristics, present a broader outlook on growth models and their applications. Although, predominantly employed in forestry research, these models transcend their initial field of use. Their robust properties and methodologies enhance accessibility for individuals without a mathematics background, ensuring easy comprehension. This paper shows some simple approach for handling with data without consistent spacing time. The newly introduced methods of estimation outlined in this paper necessitates reduced computational effort and possess the versatility to use any growth data, enhancing their applicability across diverse domains. This paper will offer some simple tools for researchers who may have limited experience in the application of most complex models. This study will help the researchers in the area of mathematical modelling.

6. Conclusion

The key focus in this study was to study about two non-linear growth models, namely, Janoschek growth model and Sloboda growth model. Unlike previous approaches which involves complex mathematics, the methods of estimation introduced in this paper are accessible, requiring no advanced mathematical knowledge. These user-friendly methods are easily implementable in any programming software, fostering broader engagement and potential for further method development. With confidence, we contend that these methods are capable to stand on par with existing techniques, thereby broadening the scope of these models in diverse growth-related fields.

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