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Some New Tripled Fixed Point Theorems in A_b -Metric Spaces

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ABSTRACT

In this paper, we establish some results on the existence and uniqueness of tripled fixed point theorems in partially ordered A_b -metric spaces. Some new tripled fixed point theorems are obtained in A_b -metric space.

Keywords-Triple fixed point, Coupled fixed point, Mixed weakly monotone property, A-metric space, b-metric space, A_b-metric with index n, A_b-metric space.

INTRODUCTION

The study of fixed point theory, some of the generalizations of metric space are 2-metric space, D-metric space, D*-metric space, G-metric space, S-metric space, Rectangular metric or metric-like space, Partial metric space, Cone metric space, etc. 1989, I.A.Bakhtin[2] introduced the concept of b-metric space. Due to the introduction of b-metric space, many generalizations of metric spaces came into existence. Recently M.Abbas, B.Ali and Y.I.Suleiman [1] introduced the concept of n-tuple metric space and studied its topological properties. M.Ughade, D.Turkoglu, S. R. Singh and R. D. Daheriya [9] introduced the notion of A_b -metric spaces as a generalized form of n-tuple metric space. Subsequently N.Mlaiki and Y.Rohen [5] obtained unique coupled common fixed point theorems in partially ordered A_b -metric spaces.

The aim of this paper is to develop unique tripled fixed point theorem and to generalize the notion of mixed weakly monotone property. We prove some unique tripled common fixed point theorems in partially ordered A_b -metric space.Before going to prove the main result, we need some basic definitions, results and examples from the literature.

PRELIMINARIES

Definition 2.1. [1].Let X be a non empty set and $n \geq 2$. A function $: X^n \rightarrow [0,\infty)$ is called an A -metric on X, if for any ξ_i, $a \in X, i = 1,2.\dots n,$ the following conditions hold.

(i) $A(\xi_1,\xi_2,\dots,\xi_{n-1},\xi_n) \geq 0$,ss

(ii) $A(\xi_1,\xi_2,\dots,\xi_{n-1},\xi_n) = 0$ if and only if $\xi_1 = \xi_2 = \dots = \xi_{n-1} = \xi_n$,

(iii) $A(\xi_1,\xi_2,\dots,\xi_{n-1},\xi_n) \leq b\left[A\left(\xi_1,\xi_1,\dots,\xi_{1_{(n-1)}},a\right) + A\left(\xi_2,\xi_2,\dots,\xi_{2_{(n-1)}},a\right) + \cdots \dots \dots \dots \dots + A\left(\xi_{n-1},\xi_{n-1},\dots,\xi_{n-1_{(n-1)}},a\right) + A\left(\xi_n,\xi_n,\dots,\xi_{n_{(n-1)}},a\right)\right]$

The pair (X,A) is called an A-metric space.

Definition 2.2 . [3]. Let X be a non empty set .A b-metric on X is a function $d : X^2 \rightarrow [0,\infty)$ such that the following conditions hold for all$x,y,z \in X$.

(i) $d(x,y) = 0$ iff $x = y$

(ii) $d(x,y) = d(y,x)$

(iii)there exists $s \geq 1$such that$d(x,z) \leq s[d(x,y) + d(y,z)]$

The pair (X,d) is called a b-metric space.

Definition 2.3. [7].Let X be a non empty set and $n \geq 2$. Suppose $b \geq 1$ is a real number. A function $A_b : X^n \rightarrow [0,\infty)$ is called an A_b - metric on X, if for any ξ_i, $a \in X, i = 1,2.\dots n,$ the following conditions hold.

(i) $A_b(\xi_1,\xi_2,\dots,\xi_{n-1},\xi_n) \geq 0$,

(ii) $A_b(\xi_1,\xi_2,\dots,\xi_{n-1},\xi_n) = 0$ if and only if $\xi_1 = \xi_2 = \dots = \xi_{n-1} = \xi_n$,

(iii) $A_b(\xi_1,\xi_2,\dots,\xi_{n-1},\xi_n) \leq b\left[A_b\left(\xi_1,\xi_1,\dots,\xi_{1_{(n-1)}},a\right) + A_b\left(\xi_2,\xi_2,\dots,\xi_{2_{(n-1)}},a\right) + \cdots \dots \dots \dots \dots + A_b\left(\xi_{n-1},\xi_{n-1},\dots,\xi_{n-1_{(n-1)}},a\right) + A_b\left(\xi_n,\xi_n,\dots,\xi_{n_{(n-1)}},a\right)\right]$

The pair (X, A_b) is called an A_b-metric space.

Note: A_b-metric space is more general than A-metric space. Moreover A-metric space is a special case of A_b-metric space with b=1. b-metric space and S_b-metric space are also special cases of A_b-metric space with n=2 and 3, respectively. Ordinary metric space and S-metric space are also special cases of A_b-metric space with b=1 and respective values of n as 2 and 3.

Lemma 2.4. [9]. Let (X, A_b) be A_b -metric space, so that $A_b: X^n \to [0,\infty)$ for somse $n \geq 2$. Then $A_b\binom{x,x,x.....x,\ y}{n-1\ times}\leq bA_b\binom{y,y,y.....y,\ x}{n-1\ times}$ for all $x, y \in X$.

Lemma 2.5. [9]. Let (X, A_b) be A_b -metric space, so that $A_b: X^n \to [0,\infty)$ for some $n \geq 2$. Then $A_b\binom{x,x,x.....x,\ \xi}{n-1\ times}\leq (n-1)bA_b\binom{x,x,x.....x,\ y}{n-1\ times}+b^2A_b\binom{y,y,y.....y,\ \xi}{n-1\ times}$ for all $x, y \in X$.

Lemma 2.6. [9]. Let (X, A_b) be A_b -metric space. Then (X^2, D_A) is A_b -metric space on $X \times X$ with D_A defined by $D_A\big((\xi 1,\vartheta 1),(\xi 2,\vartheta 2),\dots(\xi n,\vartheta n)\big) = A_b(\xi_1,\xi_2,\dots,\xi_{n-1},\xi_n)+A_b(\vartheta_1,\vartheta_2,\dots,\vartheta_{n-1},\vartheta_n)$, for all $\xi i, \vartheta i \in X, i,j = 1,2,\dots n$.

Definition 2.7. (X, A_b) be A_b -metric space. A sequence $\{\xi_k\}$ in X is said to converge to a point $x \in X$, if $A_b\binom{\xi_k,\xi_k,\xi_k.....\xi_k,\ x}{n-1\ times}\to 0\ as k \to \infty$. That is, to each $\varepsilon \geq 0$ there exist $N \in \mathbb{N}$ such that for all $k \geq N$, we have $A_b\binom{\xi_k,\xi_k,\xi_k.....\xi_k,\ x}{n-1\ times}\leq \varepsilon$ and we write $\lim_{k\to\infty} \xi_k = x$.

Lemma 2.8. [5]. Let (X, A_b) be A_b -metric space.If the sequence $\{\xi_k\} in X$ converges to a point x, then the limit x is unique.

Definition 2.9. Let (X, A_b) be A_b -metric space. A sequence $\{\xi_k\} in X$ is called a Cauchy sequence, if $A_b\binom{\xi_k,\xi_k,\xi_k.....\xi_k,\ \xi_m}{n-1\ times}\to 0\ as k,m \to \infty$. That is, to each $\varepsilon \geq 0$, there exists $N \in \mathbb{N}$ such that for all $k,m \geq N$, we have $A_b\binom{\xi_k,\xi_k,\xi_k.....\xi_k,\ \xi_m}{n-1\ times}\leq \varepsilon$.

Lemma 2.10. [5]. convergent sequence in a A_b-metric space is a Cauchy sequence.

Definition 2.11. A A_b-metric space (X, A_b) is said to be complete, if every Cauchy sequence in X is convergent.

Definition 2.12. Let $(X, \leq)$be a partially ordered set and

$F : X^3 \to X$be mappings. We say that F has the mixed monotone property on X ,if for any $x, y, z \in X$

$x_1,x_2 \in X, x_1 \leq x_2 \Longrightarrow F(x_1, y, z) \leq F(x_2, y, z)$,

$y_1,y_2 \in X, y_1 \leq y_2 \Longrightarrow F(x, y_1, z) \geq F(x, y_2, z)$,

Definition 2.13. Let $f, g : X^3 \to X$ be mappings.

An element (i) (x, y, z) is called a tripled fixed pint of f, if $x = f(x, y, z), y = f(y, x, z), z = f(z, y, x)$

An element (ii) $(x, y, z) \in X^3$is said to be a common tripled fixed pint of f and g, if $x = f(x, y, z) = g(x, y, z), y = f(y, x, z) = g(y, x, z), z = f(z, y, x) = g(z, y, x)$

Note: (x, y, z) is said to be a tripled coincident point of f And g, if $f(x, y, z) = g(x, y, z), f(y, x, z) = g(y, x, z), f(z, y, x) = g(z, y, x)$

We observe that a common tripled fixed pint of f and g is necessarily a tripled coincidence point of f and$g$

MAIN RESULTS

Before going to the main results we introduce the notion of cross product $D = A_b \times A_b \times A_b$and study its properties.

Suppose $(X \leq A)$ is a partially ordered complete A_b-metric space.

Define the partial order"$\leq$"on $X \times X \times X$ as follows .

$(x, y, z) \leq (u, v, w)$ if $x \leq u, y \geq v$ and $z \leq w$ for $(x, y, z),(u, v, w) \in X$

then "$\leq$"is a partial order on $X \times X \times X$.

Let $n \geq 2$ be a positive integer,

Define $D: (X \times X \times X)^n \to [0, \infty)$ as follows

$$D\left(\left(x_{1,}y_{1,}z_{1,}\right), \left(x_{2,}y_{2,}z_{2,}\right) \dots \dots \left(x_{n,}y_{n,}z_{n,}\right)\right) = A\left(x_{1,}, x_{2,} \dots . x_{n,}\right) + A\left(y_{1,}, y_{2,} \dots . y_{n,}\right) + A\left(z_{1,}, z_{2,} \dots . z_{n,}\right)$$

Then from lemma 2.11, $(X \times X \times X, \leq, D)$ is a A_b-metric space.

Now observe the following:

(1)A sequence $\{(x_k, y_k, z_k)\}$in $X \times X \times X \to (x, y, z) \Leftrightarrow \{x_k\} \to x, \{y_k\} \to y \text{ and } \{z_k\} \to z$

Suppose that$\{(x_k, y_k, z_k)\}$ in $X \times X \times X \to (x, y, z)$

That is, $D((x_k, y_k, z_k), (x_k, y_k, z_k) \dots \dots (x_k, y_k, z_k),(x,y,z)) \to 0 \quad as\ k \to \infty$

$$\Longrightarrow A(x_k, x_k, \dots . x_k, x) + A(y_k, y_k, \dots . y_k, y) + A(z_k, z_k, \dots . z_k, z) \to 0 \quad as\ k \to \infty$$

$$\Longrightarrow A(x_k, x_k, \dots . x_k, x) \to 0 \quad as\ kk \to \infty, A(y_k, y_k, \dots . y_k, y) \to 0 \quad as\ k \to \infty \text{ and } A(z_k, z_k, \dots . z_k, z) \to 0 \quad as\ k \to \infty$$

Therefore $\{x_k\} \to x, \{y_k\} \to y \text{ and } \{z_k\} \to z.$

Conversely suppose that $\{x_k\} \to x, \{y_k\} \to y \text{ and } \{z_k\} \to z$

That is,$A(x_k, x_k, \dots . x_k, x) \to 0 \quad as\ kA(y_k, y_k, \dots . y_k, y) \to 0 \quad as\ k \to \infty \text{ and } A(z_k, z_k, \dots . z_k, z) \to 0 \quad as\ k \to \infty$

$$\Longrightarrow A(x_k, x_k, \dots . x_k, x) + A(y_k, y_k, \dots . y_k, y) + A(z_k, z_k, \dots . z_k, z) \to 0 \quad as\ k \to \infty$$

$\Longrightarrow D((x_k, y_k, z_k), (x_k, y_k, z_k) \dots \dots (x_k, y_k, z_k),(x,y,z)) \to 0 \quad as\ k \to \infty$

Therefore$\{(x_k, y_k, z_k)\}$ in $X \times X \times X \to (x, y, z)$

(2)$\{(x_k, y_k, z_k)\}$is a Cauchy sequence in$X \times X \times X \Leftrightarrow \{x_k\}, \{y_k\} \text{ and } \{z_k\}$ are Cauchy sequence in X

Suppose that$\{(x_k, y_k, z_k)\}$ is a Cauchy sequence in $X \times X \times X$

that is, $D((x_k, y_k, z_k), (x_k, y_k, z_k) \dots (x_k, y_k, z_k)(x_m, y_m, z_m)) \to 0 \quad \text{as } k, m \to \infty$

$$\Longrightarrow A(x_k, x_k, \dots . x_k, x_m) + A(y_k, y_k, \dots . y_k, y_m) + A(z_k, z_k, \dots . z_k, z_m) \to 0 \quad as\ k, m \to \infty$$

$\Longrightarrow A(x_k, x_k, \dots . x_k, x_m) \to 0 \quad as\ k, m \to \infty,$
$A(y_k, y_k, \dots . y_k, y_m) \to 0 \quad as\ k, m \to \infty \text{ and } \quad A(z_k, z_k, \dots . z_k, z_m) \to 0 \quad as\ k, m \to \infty$

Therefore $\{x_k\}, \{y_k\} \text{ and } \{z_k\}$ are Cauchy sequence in X

Conversely suppose that$\{x_k\}, \{y_k\} \text{ and } \{z_k\}$ are Cauchy sequence in X

that is, $A(x_k, x_k, \dots . x_k, x_m) \to 0 \quad as\ k, m \to \infty, \ A(y_k, y_k, \dots . y_k, y_m) \to 0 \quad as\ k, m \to \infty \text{ and } \quad A(z_k, z_k, \dots . z_k, z_m) \to 0 \quad as\ k, m \to \infty$

$$\Longrightarrow A(x_k, x_k, \dots . x_k, x_m) + A(y_k, y_k, \dots . y_k, y_m) + A(z_k, z_k, \dots . z_k, z_m) \to 0 \quad as\ k, m \to \infty$$

$\Longrightarrow D((x_k, y_k, z_k), (x_k, y_k, z_k) \dots (x_k, y_k, z_k)(x_m, y_m, z_m)) \to 0 \quad \text{as } k, m \to \infty$

Therefore$\{(x_k, y_k, z_k)\}$ is a Cauchy sequence in $X \times X \times X$

(3)$X$is complete$\Leftrightarrow X \times X \times X$ is complete

Suppose$\{x_k\}, \{y_k\} \text{ and } \{z_k\}$ are complete in X

that is,$\{x_k\} \to x, \{y_k\} \to y \text{ and } \{z_k\} \to z$

$$\Longrightarrow A(x_k, x_k, \dots . x_k, x) \to 0 \quad as\ k \to \infty,$$
$$A(y_k, y_k, \dots . y_k, y) \to 0 \quad as\ k \to \infty \text{ and}$$

$$A(z_k, z_k, \ldots . z_k, z) \to 0 \quad as\ k \to \infty$$

$$\Longrightarrow A(x_k, x_k, \ldots . x_k, x) + A(y_k, y_k, \ldots . y_k, y) + A(z_k, z_k, \ldots . z_k, z) \to 0 \quad as\ k \to \infty$$

$\Longrightarrow D((x_k, y_k, z_k), (x_k, y_k, z_k) \ldots \ldots (x_k, y_k, z_k), (x,y,z)) \to 0 \quad as\ k \to \infty$

thus $\{(x_k, y_k, z_k)\}$ is complete in $X \times X \times X$

Conversely suppose$\{(x_k, y_k, z_k)\}$ is complete in $X \times X \times X$

that is, $\{(x_k, y_k, z_k)\} \to (x, y, z)$

$\Longrightarrow D((x_k, y_k, z_k), (x_k, y_k, z_k) \ldots \ldots (x_k, y_k, z_k), (x,y,z)) \to 0 \quad as\ k \to \infty$

$$\Longrightarrow A(x_k, x_k, \ldots . x_k, x) + A(y_k, y_k, \ldots . y_k, y) + A(z_k, z_k, \ldots . z_k, z) \to 0 \quad as\ k \to \infty$$

$$\Longrightarrow A(x_k, x_k, \ldots . x_k, x) \to 0 \quad as\ k \to \infty,$$

$$A(y_k, y_k, \ldots . y_k, y) \to 0 \quad as\ k \to \infty\ and$$

$$A(z_k, z_k, \ldots . z_k, z) \to 0 \quad as\ k \to \infty$$

therefore $\{x_k\} \to x, \{y_k\} \to y\ and\ \{z_k\} \to z.$

Now we prove our first main result

Theorem 3.1.Let $(X, \leq, A)$ be a partially ordered complete A_b $-$metric space and $f: X^3 \to X$ such that (i) f has mixed weakly monotone property on X and there exist $x_0, y_0, z_0\ in\ X$ such that

$x_0 \leq f(x_0, y_0, z_0)$,$f(y_0, x_0, z_0) \leq y_0$,and $z_0 \leq f(z_0, y_0, x_0)$

(ii) there is an α such that $\alpha b^2((n-1)b+1) < 1$ and

A(f(x,y,z),f(x,y,z),....f(x,y,z),f(u,v,w))+A(f(y,x,z),f(y,x,z),....f(y,x,z),f(v,u,w) +A(f(z,y,x),f(z,y,x),....f(z,y,x),f(w,v,u))

$\leq \alpha$max{D((x,y,z),(x,y,z),...(x,y,z),(u,v,w)), D((x,y,z),(x,y,z),...(x,y,z),(f(x,y,z),f(y,x,z),f(z,y,x))), D((u,v,w),(u,v,w),...,(u,v,w),(f(u,v,w),f(v,u,w)),f(w,v,u)),D((x,y,z),(x,y,z),...(x,y,z),(f(u,v,w),f(v,u,w),f(w,v,u))),D((u,v ,w),(u,v,w),...,(u,v,w),(f(x,y,z),f(y,x,z),f(z,y,x)))} (3.1.1)

for all $x, y, z, u, v, w \in X$ with $x \leq$u,$y \geq v$and $z \leq w$ (iii) $f$is continuous or X has the following properties.

(a) if $\{x_k\}$ is an increasing sequence with

$x_k \to x$ then $x_k \leq x$ for all $k \in$ N

(b) if $\{y_k\}$ is a decreasing sequence with $y_k \to y$, then $y \leq y_k$ for all $k \in$ N

(c) if $\{z_k\}$ is an increasing sequence with

$z_k \to z$ then $z_k \leq z$ for all $k \in$ N

Then $f$has a Tripled fixed point in X

Proof.Let (x_0, y_0, z_0) be a given point in $X \times X \times X$,satisfying (i).

Write $x_1 = f(x_0, y_0, z_0)$, $y_1 = f(y_0, x_0, z_0)$, $z_1 = f(z_0, y_0, x_0)$ and $x_2 = f(x_1, y_1, z_1)$, $y_2 = f(y_1, x_1, z_1), z_2 = f(z_1, y_1, x_1)$

Define the sequences $\{x_k\}, \{y_k\}\ and\ \{z_k\}$inductively

$x_{2k+1} = f(x_{2k}, y_{2k}, z_{2k}), y_{2k+1} = f(y_{2k}, x_{2k}, z_{2k}), z_{2k+1} = f(z_{2k}, y_{2k}, x_{2k})$

$x_{2k+2} = f(x_{2k+1}, y_{2k+1}, z_{2k+1}), y_{2k+2} = f(y_{2k+1}, x_{2k+1}, z_{2k+1}), z_{2k+2} = f(z_{2k+1}, y_{2k+1}, x_{2k+1})$for all $k \in$ N (3.1.2)

Since $x_0 \leq f(x_0, y_0, z_0)$,$y_0 \geq f(y_0, x_0, z_0)$and $z_0 \leq f(z_0, y_0, x_0)$

and since f has mixed weakly monotone property, we get

$x_1 = f(x_0, y_0, z_0) \leq f(f(x_0, y_0, z_0), f(y_0, x_0, z_0), f(z_0, y_0, x_0)) = f(x_1, y_1, z_1) = x_2 \Longrightarrow x_1 \leq x_2,$

$$x_2 = f(x_1, y_1, z_1) \leq f(f(x_1, y_1, z_1), f(y_1, x_1, z_1), f(z_1, y_1, x_1)) = f(x_2, y_2, z_2) = x_3 \Longrightarrow x_2 \leq x_3$$

and $y_1 = f(y_0, x_0, z_0) \geq f(f(y_0, x_0, z_0), f(x_0, y_0, z_0), f(z_0, y_0, x_0)) = y_2 \Longrightarrow y_1 \geq y_2$,

$$y_2 = f(y_1, x_1, z_1 \geq f(f(y_1, x_1, z_1), f(x_1, y_1, z_1), f(z_1, y_1, x_1)) = y_3 \Longrightarrow y_2 \geq y_3$$

also

$$z_1 = f(z_0, y_0, x_0) \leq f(f(z_0, y_0, x_0), f(y_0, x_0, z_0), f(x_0, y_0, z_0)) = f(z_1, y_1, x_1) = z_2 \Longrightarrow z_1 \leq z_2,$$

$$z_2 = f(z_1, y_1, x_1 \leq f(f(z_1, y_1, x_1), f(y_1, x_1, z_1), f(x_1, y_1, z_1)) = f(z_2, y_2, x_2) = z_3 \Longrightarrow z_2 \leq z_3$$

By induction,

$x_0 \leq x_1 \leq x_2 \leq$......$\leq x_k \leq x_{2k+1} \leq$........

$y_0 \geq y_1 \geq y_2 \geq$......$\geq y_k \geq y_{2k+1} \geq$.......... (3.1.3)

$z_0 \leq z_1 \leq z_2 \leq$......$\leq z_k \leq z_{2k+1} \leq$........ for all $k \in$ N

Define $D_k: X^3 \longrightarrow X$ by

$$D_k = \mathrm{D}\begin{pmatrix} (x_k, y_k, z_{k)},(x_k,y_k,z_{k)}, \ldots . (x_k, y_k, z_k), (x_{k+1}, y_{k+1}, z_{k+1}), \\ (n-1\ times) \qquad ⬚ \end{pmatrix}$$

$$= A_b\begin{pmatrix} x_{k,,x_k}, \ldots, x_k, x_{k+1} \\ (n-1\ times) \quad ⬚ \end{pmatrix} + A_b\begin{pmatrix} y_{k,,y_k}, \ldots, y_k, y_{k+1} \\ (n-1\ times) \quad ⬚ \end{pmatrix} + A_b\begin{pmatrix} z_{k,,z_k}, \ldots, z_k, z_{k+1} \\ (n-1\ times) \quad ⬚ \end{pmatrix}$$

for k=0,1,2....

D_{2k+1}= A_b (x_{2k+1},x_{2k+1},....x_{2k+1},x_{2k+2})+ A_b (y_{2k+1},$y_{2k+1}, \ldots . y_{2k+1}, y_{2k+2}$)+ A_b (z_{2k+1},$z_{2k+1}, \ldots . z_{2k+1}, z_{2k+2}$)
=
$A_b(f(x_{2k}, y_{2k}, z_{2k})$,$f(x_{2k}, y_{2k}, z_{2k})$...$f(x_{2k}, y_{2k}, z_{2k})$,$f(x_{2k+1}, y_{2k+1}, z_{2k+1}))$ + $A_b(f(y_{2k}, x_{2k}, z_{2k})$,
$f(y_{2k}, x_{2k}, z_{2k})$,...$f(y_{2k}, x_{2k}, z_{2k})$,$f(y_{2k+1}, x_{2k+1}, z_{2k+1})$)+ $A_b(f(z_{2k}, y_{2k}, x_{2k})$,$f(z_{2k}, y_{2k}, x_{2k})$,...$f(z_{2k}, y_{2k}, x_{2k})$,,
$f(z_{2k+1}, y_{2k+1}, x_{2k+1}))$
$\leq \alpha$ max {
D((x_{2k}, y_{2k}, z_{2k}),(x_{2k}, y_{2k}, z_{2k}),$\ldots .(x_{2k}, y_{2k}, z_{2k}), (x_{2k+1}, y_{2k+1}, z_{2k+1})$),D($(x_{2k}, y_{2k}, z_{2k})$,$(x_{2k}, y_{2k}, z_{2k})$,,...
(x_{2k}, y_{2k}, z_{2k}),($f(x_{2k}, y_{2k}, z_{2k})$,$f(y_{2k}, x_{2k}, z_{2k}), f(z_{2k}, y_{2k}, x_{2k})$)),
D($(x_{2k+1}, y_{2k+1}, z_{2k+1}), (x_{2k+1}, y_{2k+1}, z_{2k+1})$,...$(x_{2k+1}, y_{2k+1}, z_{2k+1})$,($f(x_{2k+1}, y_{2k+1}, z_{2k+1})$,
$f(y_{2k+1}, x_{2k+1}, z_{2k+1})$,$f(z_{2k+1}, y_{2k+1}, x_{2k+1})$)),D($(x_{2k}, y_{2k}, z_{2k})$,$(x_{2k}, y_{2k}, z_{2k})$,,...$(x_{2k}, y_{2k}, z_{2k})$,(
$f(x_{2k+1}, y_{2k+1}, z_{2k+1}), f(y_{2k+1}, x_{2k+1}, z_{2k+1}), f(z_{2k+1}, y_{2k+1}, x_{2k+1})$)),D(
$(x_{2k+1}, y_{2k+1}, z_{2k+1}), (x_{2k+1}, y_{2k+1}, z_{2k+1})$,...$(x_{2k+1}, y_{2k+1}, z_{2k+1})$, ($f(x_{2k}, y_{2k}, z_{2k}), f(y_{2k}, x_{2k}, z_{2k})$,
$f(z_{2k}, y_{2k}, x_{2k})$))}

From (3.1.2)

$D_{2k+1} \leq \alpha$ max { D((x_{2k}, y_{2k}, z_{2k}),(x_{2k}, y_{2k}, z_{2k}),$\ldots .(x_{2k}, y_{2k}, z_{2k}), (x_{2k+1}, y_{2k+1}, z_{2k+1})$),
D((x_{2k}, y_{2k}, z_{2k}),(x_{2k}, y_{2k}, z_{2k}),,...(x_{2k}, y_{2k}, z_{2k}), , $(x_{2k+1}, y_{2k+1}, z_{2k+1})$),
D($(x_{2k+1}, y_{2k+1}, z_{2k+1}), (x_{2k+1}, y_{2k+1}, z_{2k+1})$,...$(x_{2k+1}, y_{2k+1}, z_{2k+1}), (x_{2k+2}, y_{2k+2}, z_{2k+2})$),
D((x_{2k}, y_{2k}, z_{2k}),(x_{2k}, y_{2k}, z_{2k}),$\ldots .(x_{2k}, y_{2k}, z_{2k})$,
$(x_{2k+2}, y_{2k+2}, z_{2k+2})$),D($(x_{2k+1}, y_{2k+1}, z_{2k+1}), (x_{2k+1}, y_{2k+1}, z_{2k+1})$,...$(x_{2k+1}, y_{2k+1}, z_{2k+1})$,
$(x_{2k+1}, y_{2k+1}, z_{2k+1})$)} (3.1.4)

= α max { D((x_{2k}, y_{2k}, z_{2k}),(x_{2k}, y_{2k}, z_{2k}),$\ldots .(x_{2k}, y_{2k}, z_{2k}), (x_{2k+1}, y_{2k+1}, z_{2k+1})$),
D((x_{2k}, y_{2k}, z_{2k}),(x_{2k}, y_{2k}, z_{2k}),,...(x_{2k}, y_{2k}, z_{2k}), , $(x_{2k+1}, y_{2k+1}, z_{2k+1})$),
D($(x_{2k+1}, y_{2k+1}, z_{2k+1}), (x_{2k+1}, y_{2k+1}, z_{2k+1})$,...$(x_{2k+1}, y_{2k+1}, z_{2k+1}), (x_{2k+2}, y_{2k+2}, z_{2k+2})$),
D((x_{2k}, y_{2k}, z_{2k}),(x_{2k}, y_{2k}, z_{2k}),$\ldots .(x_{2k}, y_{2k}, z_{2k}), (x_{2k+2}, y_{2k+2}, z_{2k+2})$)}

From lemma 2.10,

$D_{2k+1} \leq \alpha$ max { D((x_{2k}, y_{2k}, z_{2k}),(x_{2k}, y_{2k}, z_{2k}),$\ldots .(x_{2k}, y_{2k}, z_{2k}), (x_{2k+1}, y_{2k+1}, z_{2k+1})$),
D((x_{2k}, y_{2k}, z_{2k}),(x_{2k}, y_{2k}, z_{2k}),,...(x_{2k}, y_{2k}, z_{2k}), , $(x_{2k+1}, y_{2k+1}, z_{2k+1})$),
D($(x_{2k+1}, y_{2k+1}, z_{2k+1}), (x_{2k+1}, y_{2k+1}, z_{2k+1})$,...$(x_{2k+1}, y_{2k+1}, z_{2k+1}), (x_{2k+2}, y_{2k+2}, z_{2k+2})$), $(n-1)b$
D((x_{2k}, y_{2k}, z_{2k}),(x_{2k}, y_{2k}, z_{2k}),$\ldots .(x_{2k}, y_{2k}, z_{2k}), (x_{2k+1}, y_{2k+1}, z_{2k+1})$)
+b^2D($(x_{2k+1}, y_{2k+1}, z_{2k+1}), (x_{2k+1}, y_{2k+1}, z_{2k+1})$,...$(x_{2k+1}, y_{2k+1}, z_{2k+1}), (x_{2k+2}, y_{2k+2}, z_{2k+2})$)}

$\leq \alpha(n-1)b$ D((x_{2k}, y_{2k}, z_{2k}) , (x_{2k}, y_{2k}, z_{2k}) , $\ldots .(x_{2k}, y_{2k}, z_{2k}), (x_{2k+1}, y_{2k+1}, z_{2k+1})$)
+b^2D($(x_{2k+1}, y_{2k+1}, z_{2k+1}), (x_{2k+1}, y_{2k+1}, z_{2k+1})$,...$(x_{2k+1}, y_{2k+1}, z_{2k+1}), (x_{2k+2}, y_{2k+2}, z_{2k+2})$)

$= \alpha(n-1)b\,[A_b\,(x_{2k},x_{2k},\dots x_{2k},x_{2k+1})+A_b\,(y_{2k},y_{2k},\dots y_{2k},y_{2k+1})+A_b\,(z_{2k},z_{2k},\dots z_{2k},z_{2k+1})] + b^2\,[A_b\,(x_{2k+1},x_{2k+1},\dots x_{2k+1},x_{2k+2})+A_b\,(y_{2k+1},y_{2k+1},\dots y_{2k+1},y_{2k+2})+A_b\,(z_{2k+1},z_{2k+1},\dots z_{2k+1},z_{2k+2})]$
(3.1.5)

Therefore
$D_{2k+1} \le \alpha(n-1)b\,[A_b\,(x_{2k},x_{2k},\dots x_{2k},x_{2k+1})+A_b\,(y_{2k},y_{2k},\dots y_{2k},y_{2k+1})+A_b\,(z_{2k},z_{2k},\dots z_{2k},z_{2k+1})] + b^2\,[A_b\,(x_{2k+1},x_{2k+1},\dots x_{2k+1},x_{2k+2})+A_b\,(y_{2k+1},y_{2k+1},\dots y_{2k+1},y_{2k+2})+A_b\,(z_{2k+1},z_{2k+1},\dots z_{2k+1},z_{2k+2})]$
(3.1.6)

$$\Rightarrow (1-\alpha\, b^2)D_{2k+1} \le \alpha\,(n-1)bD_{2k}$$
$$\Rightarrow D_{2k+1} \le \frac{\alpha\,(n-1)b}{(1-\alpha\, b^2)} D_{2k}$$

Put $\beta = \frac{\alpha\,(n-1)b}{(1-\alpha\, b^2)}$, then $0 \le \beta < 1$(3.1.7)
From (3.1.7), $D_{2k+1} \le \beta D_{2k}$
Similarly we can show that $D_{2k+2} \le \beta D_{2k+1}$ \ for k=0,1,2,...
Hence $D_{k+1} \le \beta D_k$
Therefore
$D_{k+1} \le \beta^{k+1} D_0$ (3.1.8)

$$D_{k,l} = \mathrm{D}\begin{pmatrix}(x_k,y_k,z_k),(x_k,y_k,z_k)\dots(x_k,y_k,z_k),(x_l,y_l,z_l),\\ (n-1\ times)\end{pmatrix}$$
$$= A_b\begin{pmatrix}x_k,x_k,\dots,x_k,x_l\\(n-1\ times)\end{pmatrix} + A_b\begin{pmatrix}y_k,y_k,\dots,y_k,y_l\\(n-1\ times)\end{pmatrix} + A_b\begin{pmatrix}z_k,z_k,\dots,z_k,z_l\\(n-1\ times)\end{pmatrix}$$

Now we have to show that $D_{k,l}$ is a Cauchy sequence
By lemma 2.10, for all $k,m \in N, k \le m$
we have

$D_{k+1,m+1} = A_b\begin{pmatrix}x_{k+1},x_{k+1},\dots,x_{k+1},x_{m+1}\\(n-1\ times)\end{pmatrix} + A_b\begin{pmatrix}y_{k+1},y_{k+1},\dots,y_{k+1},y_{m+1}\\(n-1\ times)\end{pmatrix} + A_b\begin{pmatrix}z_{k+1},z_{k+1},\dots,z_{k+1},z_{m+1}\\(n-1\ times)\end{pmatrix}$
$\le (n-1)b\,[A_b\begin{pmatrix}x_{k+1},x_{k+1},\dots,x_{k+1},x_{k+2}\\(n-1\ times)\end{pmatrix} + A_b\begin{pmatrix}y_{k+1},y_{k+1},\dots,y_{k+1},y_{k+2}\\(n-1\ times)\end{pmatrix} + A_b\begin{pmatrix}z_{k+1},z_{k+1},\dots,z_{k+1},z_{k+2}\\(n-1\ times)\end{pmatrix}]$
$+b^2\,[A_b\begin{pmatrix}x_{k+2},x_{k+2},\dots,x_{k+2},x_{m+1}\\n-1\ times\end{pmatrix} + A_b\begin{pmatrix}y_{k+2},y_{k+2},\dots,y_{k+2},y_{m+1}\\n-1\ times\end{pmatrix} + A_b\begin{pmatrix}z_{k+2},z_{k+2},\dots,z_{k+2},z_{m+1}\\n-1\ times\end{pmatrix}]$
$= (n-1)bD_{k+1} + b^2(n-1)b[\,A_b\begin{pmatrix}x_{k+2},x_{k+2},\dots,x_{k+2},x_{k+3}\\n-1\ times\end{pmatrix} + A_b\begin{pmatrix}y_{k+2},y_{k+2},\dots,y_{k+2},y_{k+3}\\n-1\ times\end{pmatrix} +$
$A_b\begin{pmatrix}z_{k+2},z_{k+2},\dots,z_{k+2},z_{k+3}\\n-1\ times\end{pmatrix}]$
$+b^2b^2\,[A_b\begin{pmatrix}x_{k+3},x_{k+3},\dots,x_{k+3},x_{m+1}\\n-1\ times\end{pmatrix} + A_b\begin{pmatrix}y_{k+3},y_{k+3},\dots,y_{k+3},y_{m+1}\\n-1\ times\end{pmatrix} + A_b\begin{pmatrix}z_{k+3},z_{k+3},\dots,z_{k+3},z_{m+1}\\n-1\ times\end{pmatrix}]$
$= (n-1)bD_{k+1} + b^3(n-1)D_{k+2} + b^5(n-1)D_{k+3} + b^{2(m-k)-3}(n-1))\,[A_b\begin{pmatrix}x_{m-1},x_{m-1},\dots,x_{m-1},x_m\\n-1\ times\end{pmatrix} +$
$A_b\begin{pmatrix}y_{m-1},y_{m-1},\dots,y_{m-1},y_m\\n-1\ times\end{pmatrix} + A_b\begin{pmatrix}z_{m-1},z_{m-1},\dots,z_{m-1},z_m\\n-1\ times\end{pmatrix}] + b^{2(m-k)-1}(n-1))\,[A_b\begin{pmatrix}x_m,x_m,\dots,x_m,x_{m+1}\\n-1\ times\end{pmatrix} +$
$A_b\begin{pmatrix}y_m,y_m,\dots,y_m,y_{m+1}\\n-1\ times\end{pmatrix} + A_b\begin{pmatrix}z_m,z_m,\dots,z_m,z_{m+1}\\n-1\ times\end{pmatrix}$

From (3.1.8)
$D_{k+1,m+1} \le (n-1)b[\beta^{k+1} + b^2\beta^{k+2} + b^4\beta^{k+3} + \cdots \dots + b^{2(m-k)-2}\beta^m]\,D_0$

$$\Rightarrow D_{k+1,m+1} \le (n-1)b\beta^{k+1}[1 + b^2\beta + (b^2\beta)^2 + \cdots + (b^2\beta)^{(m-k-1)}]D_0$$
$$= (n-1)b\beta^{k+1}[1+\gamma+\gamma^2+\cdots+\gamma^{(m-k-1)}]D_0$$
$$\le (n-1)b\beta^{k+1}\frac{1}{1-\gamma}\,D_0$$

Where $\gamma = b^2\beta$
Hence for all $k,m \in N, k \le m$, *we have*

$$D_{k,m} = A_b\begin{pmatrix}x_k,x_k,\dots,x_k,x_m\\n-1\ times\end{pmatrix} + A_b\begin{pmatrix}y_k,y_k,\dots,y_k,y_m\\n-1\ times\end{pmatrix} + A_b\begin{pmatrix}z_k,z_k,\dots,z_k,z_m\\n-1\ times\end{pmatrix} \le (n-1)b\beta^k\frac{1}{1-\gamma}\,D_0$$

Since $0 \le \gamma = \alpha b^2((n-1)b+1) < 1$, we have

$$\lim_{k,m\to\infty} A_b\begin{pmatrix}x_k,x_k,\dots,x_k,x_m\\n-1\ times\end{pmatrix} + A_b\begin{pmatrix}y_k,y_k,\dots,y_k,y_m\\n-1\ times\end{pmatrix} + A_b\begin{pmatrix}z_k,z_k,\dots,z_k,z_m\\n-1\ times\end{pmatrix} = 0$$

That is,

$$\lim_{k,m\to\infty} A_b\begin{pmatrix}x_k,x_k,\dots,x_k,x_m\\n-1\ times\end{pmatrix} = \lim_{k,m\to\infty} A_b\begin{pmatrix}y_k,y_k,\dots,y_k,y_m\\n-1\ times\end{pmatrix} = \lim_{k,m\to\infty} A_b\begin{pmatrix}z_k,z_k,\dots,z_k,z_m\\n-1\ times\end{pmatrix} = 0$$

Therefore $\{x_k\}, \{y_k\}\ and\ \{z_k\}$are Cauchy sequences in X.

By the completeness of X, there exists $x, y, z \in X$ such that $x_k \to x$, $y_k \to y$ and $z_k \to z$ as $k \to \infty$

Therefore $D_{k,l}$ is a Cauchy sequence.

Now we show that (x, y, z) is a tripled fixed point of f

Suppose that f is continuous, we have

$$x = \lim_{k\to\infty} x_{2k+1} = \lim_{k\to\infty} f(x_{2k}, y_{2k}, z_{2k}) = f\left(\lim_{k\to\infty} x_{2k}, \lim_{k\to\infty} y_{2k}, \lim_{k\to\infty} z_{2k}\right) = f(x, y, z)$$

$$y = \lim_{k\to\infty} y_{2k+1} = \lim_{k\to\infty} f(y_{2k}, x_{2k}, z_{2k}) = f\left(\lim_{k\to\infty} y_{2k}, \lim_{k\to\infty} x_{2k}, \lim_{k\to\infty} z_{2k}\right) = f(y, x, z)$$

and

$$z = \lim_{k\to\infty} z_{2k+1} = \lim_{k\to\infty} f(z_{2k}, y_{2k}, x_{2k}) = f\left(\lim_{k\to\infty} z_{2k}, \lim_{k\to\infty} y_{2k}, \lim_{k\to\infty} x_{2k}\right) = f(z, y, x)$$

From (3.1.1), we have

A_b(x,x,....x,f(x,y,z))+A(y,y,....y,f(y,x,z))+A(z,z,....z,f(z,y,x))

= A_b(f(x,y,z),f(x,y,z),....f(x,y,z),f(x,y,z))+ A_b(f(y,x,z),f(y,x,z),....f(y,x,z),f(y,x,z))+ A_b(f(z,y,z),f(z,y,x),....f(z,y,x),f(z,y,x))

$\leq$ α max{
D((x,y,z),(x,y,z),...(x,y,z),(x,y,z)),D((x,y,z),(x,y,z),...(x,y,z),(f(x,y,z),f(y,x,z),f(z,y,x))),D((x,y,z),(x,y,z),...(x,y,z),(f(x,y,z),
f(y,x,z),f(z,y,x))),D((x,y,z),(x,y,z),...(x,y,z),(f(x,y,z),f(y,x,z),f(z,y,x))),D((x,y,z),(x,y,z),...(x,y,z),(f(x,y,z),f(y,x,z),f(z,y,x)))
}

= α max { 0, D((x,y,z),(x,y,z),...(x,y,z),(f(x,y,z),f(y,x,z),f(z,y,x)))}

= α{ D((x,y,z),(x,y,z),...(x,y,z),(f(x,y,z),f(y,x,z),f(z,y,x)))}

$\leq$ αb((f(x,y,z),f(y,x,z),f(z,y,x)),(f(x,y,z),f(y,x,z),f(z,y,x)),...(f(x,y,z),f(y,x,z),f(z,y,x)),(x,y,z))

Since $\alpha b < 1$, we have (f(x,y,z),f(y,x,z),f(z,y,x))=(x,y,z)

$\Longrightarrow$f(x,y,z)=x, f(y,x,z)=y and f(z,y,x)=z

Therefore (x,y,z) is a tripled fixed point of f.

Suppose X satisfies (a) and (b), by (3.1.3), we get $x_k \leq x$, $y_k \geq y$ and $z_k \leq z$ for all $k \in N$

applying lemmas 2.10 and 2.11, we have

D((x,y,z),(x,y,z),...(x,y,z),(f(x,y,z),f(y,x,z),f(z,y,x))

$\leq b(n-1)$D((x,y,z),(x,y,z),...(x,y,z),(x_{2k+2},y_{2k+2},z_{2k+2}))+b^2D((x_{2k+2},y_{2k+2}, z_{2k+2}),(x_{2k+2},y_{2k+2}, z_{2k+2}),....(x_{2k+2}, y_{2k+2}, z_{2k+2}),(f(x,y,z),f(y,x,z),f(z,y,x)))

$= b(n-1)$

D((x,y,z),(x,y,z),...(x,y,z),(x_{2k+2},y_{2k+2}, z_{2k+2}))

+b^2D((f(x_{2k+1},y_{2k+1}, z_{2k+1}),f(y_{2k+1},x_{2k+1}, z_{2k+1}),f(z_{2k+1},y_{2k+1}, x_{2k+1})),(f(x_{2k+1},y_{2k+1}, z_{2k+1}),f(y_{2k+1}, x_{2k+1}, z_{2k+1}),f(z_{2k+1},y_{2k+1}, x_{2k+1})),.....(f(x_{2k+1},y_{2k+1}, z_{2k+1}),f(y_{2k+1},x_{2k+1}, z_{2k+1}),f(z_{2k+1},y_{2k+1}, x_{2k+1})),(f(x,y,z),f(y,x,z),f(z,y,x)))

$\Longrightarrow$D((x,y,z),(x,y,z),...(x,y,z),(f(x,y,z),f(y,x,z),f(z,y,x)))$\leq$ $b(n-1)$ [A_b(x,x,...x,x_{2k+2})+ A_b(y,y,...y,y_{2k+2}) + A_b(z,z,...z,z_{2k+2})]+b^2 A_b[f(x_{2k+1},y_{2k+1}, z_{2k+1}),f(x_{2k+1},y_{2k+1}, z_{2k+1}),.....f(x_{2k+1},y_{2k+1}, z_{2k+1}),f(x,y,z)]+b^2 A_b[f(y_{2k+1},x_{2k+1}, z_{2k+1}),f(y_{2k+1},x_{2k+1}, z_{2k+1}).....f(y_{2k+1},x_{2k+1}, z_{2k+1}),(f(y,x,z))]+b^2 A_b[f(z_{2k+1},y_{2k+1}, x_{2k+1}),f(z_{2k+1},y_{2k+1}, x_{2k+1}),.....f(z_{2k+1},y_{2k+1}, x_{2k+1}),(f(z,y,x))] (3.1.9)

By using (3.1.1), we get

A_b[f(x_{2k+1},y_{2k+1}, z_{2k+1}),f(x_{2k+1},y_{2k+1}, z_{2k+1}),.....f(x_{2k+1},y_{2k+1}, z_{2k+1}),f(x,y,z)]+ A_b
[f(y_{2k+1},x_{2k+1}, z_{2k+1}),f(y_{2k+1},x_{2k+1}, z_{2k+1}).....f(y_{2k+1},x_{2k+1}, z_{2k+1}),(f(y,x,z))]+ A_b
[f(z_{2k+1},y_{2k+1}, x_{2k+1}),f(z_{2k+1},y_{2k+1}, x_{2k+1}),.....f(z_{2k+1},y_{2k+1}, x_{2k+1}),(f(z,y,x))]

$\leq$ α max {
(D(x_{2k+1},y_{2k+1}, z_{2k+1}),(x_{2k+1},y_{2k+1}, z_{2k+1}),.....(x_{2k+1},y_{2k+1}, z_{2k+1}),(x,y,z)),(D(x_{2k+1},y_{2k+1}, z_{2k+1}),(x_{2k+1}, y_{2k+1}, z_{2k+1}),.....(x_{2k+1},y_{2k+1}, z_{2k+1}),(f(x_{2k+1},y_{2k+1}, z_{2k+1}),f(y_{2k+1},x_{2k+1}, z_{2k+1}),f(z_{2k+1},y_{2k+1}, x_{2k+1}))),D((x,y,z),(x,y,z),...(x,y,z),(f(x,y,z),f(y,x,z),f(z,y.x))),(D(x_{2k+1},y_{2k+1}, z_{2k+1}),(x_{2k+1},y_{2k+1}, z_{2k+1}),.....(x_{2k+1}, y_{2k+1}, z_{2k+1}),(f(x,y,z),f(y,x,z),f(z,y,x))),D((x,y,z),(x,y,z),...(x,y,z),,(f(x_{2k+1},y_{2k+1}, z_{2k+1}),f(y_{2k+1},x_{2k+1}, z_{2k+1}), f(z_{2k+1},y_{2k+1}, x_{2k+1})))}

= α max
{(D(x_{2k+1},y_{2k+1}, z_{2k+1}),(x_{2k+1},y_{2k+1}, z_{2k+1}),.....(x_{2k+1},y_{2k+1}, z_{2k+1}),(x,y,z)),(D(x_{2k+1},y_{2k+1}, z_{2k+1}),(x_{2k+1}, y_{2k+1}, z_{2k+1}),.....(x_{2k+1},y_{2k+1}, z_{2k+1}),($x_{2k+2}$$y_{2k+2}$, z_{2k+2})),
D((x,y,z),(x,y,z),...(x,y,z),(f(x,y,z),f(y,x,z),f(z,y,x))),(D(x_{2k+1},y_{2k+1}, z_{2k+1}),(x_{2k+1},y_{2k+1}, z_{2k+1}),.....(x_{2k+1}, y_{2k+1}, z_{2k+1}),(f(x,y,z),f(y,x,z),f(z,y,x))),D((x,y,z),(x,y,z),...(x,y,z),(x_{2k+2},y_{2k+2}, z_{2k+2}))}

From (3.1.6) and (3.1.7)

D((x,y,z),(x,y,z),...(x,y,z),(f(x,y,z),f(y,x,z),f(z,y,x)))≤ $b(n-1)[$ *A_b(x,x,...x,x_{2k+2})+ A_b(y,y,...y,y_{2k+2})* $+$ *A_b(z,z,...z,z_{2k+2})]+$b^2\alpha$ max { (D(x_{2k+1},y_{2k+1}, z_{2k+1}),(x_{2k+1},y_{2k+1}, z_{2k+1}),.....(x_{2k+1},y_{2k+1}, z_{2k+1}),(x,y,z)),*

(D(x_{2k+1},y_{2k+1}, z_{2k+1}),(x_{2k+1},y_{2k+1}, z_{2k+1}),.....(x_{2k+1},y_{2k+1}, z_{2k+1}),($x_{2k+2}$$y_{2k+2}$, z_{2k+2})),D((x,y,z),(x,y,z),...(x,y, z),(f(x,y,z),f(y,x,z),f(z,y,x))),(D(x_{2k+1},y_{2k+1}, z_{2k+1}),(x_{2k+1},y_{2k+1}, z_{2k+1}),.....(x_{2k+1},y_{2k+1}, z_{2k+1}), (f(x,y,z),f(y,x,z),f(z,y,x))), D((x,y,z),(x,y,z),...(x,y,z),($x_{2k+2}$$y_{2k+2}$, z_{2k+2})\}

Taking the limit as $k\to\infty$, in (3.1.8), we obtain

D((x,y,z),(x,y,z),...(x,y,z),(f(x,y,z),f(y,x,z),f(z,y,x)))≤ $b(n-1)$ *[A(x,x,...x,x)+A(y,y,...y,y)+A(z,z,...z,z)]+$b^2\alpha$ max { D((x,y,z),(x,y,z),...(x,y,z),(x,y,z)),D((x,y,z),(x,y,z),...(x,y,z),(x,y,z)),D((x,y,z),(x,y,z),...(x,y,z),(f(x,y,z),f(y,x,z),f(z,y,x))),D((x,y,z),(x,y,z),...(x,y,z),(f(x,y,z),f(y,x,z),f(z,y,x))),D((x,y,z),(x,y,z),...(x,y,z),(x,y,z))}*

=$b^2\alpha$ D((x,y,z),(x,y,z),...(x,y,z),(f(x,y,z),f(y,x,z),f(z,y,x)))

Since $b^2\alpha$<1, we have

D((x,y,z),(x,y,z),...(x,y,z),(f(x,y,z),f(y,x,z),f(z,y,x)))=0

Therefore (f(x,y,z),f(y,x,z),f(z,y,x))=(x,y,z)

That is, f(x,y,z)=x , f(y,x,z)=y and f(z,y,x)=z

Therefore (x,y,z) is a tripled fixed point of f .

Theorem 3.2.Let $(X,\leq,A)$ be a partially ordered complete A_b −metric space and $f,g\colon X^3\to X$ such that (i) the pair (f,g) has mixed weakly monotone property on X and there exist x_0,y_0,z_0 in X such that

$x_0\leq f(x_0,y_0,z_0)$, $f(y_0,x_0,z_0)\leq y_0$,*and* $z_0\leq f(z_0,y_0,x_0)$ *or* $x_0\leq g(x_0,y_0,z_0)$, $g(y_0,x_0,z_0)\leq y_0$,*and* $z_0\leq g(z_0,y_0,x_0)$

(ii) there is an α such that $\alpha b^2((n-1)b+1)<1$ *and*

A(f(x,y,z),f(x,y,z),....f(x,y,z),g(u,v,w))+A(f(y,x,z),f(y,x,z),....f(y,x,z),g(v,u,w) +A(f(z,y,x),f(z,y,x),....f(z,y,x),g(w,v,u))

≤ α max{D((x,y,z),(x,y,z),...(x,y,z),(u,v,w)), D((x,y,z),(x,y,z),...(x,y,z),(f(x,y,z),f(y,x,z),f(z,y,x))), D((u,v,w),(u,v,w),...,(u,v,w),(g(u,v,w),g(v,u,w)),g(w,v,u)),D((x,y,z),(x,y,z),...(x,y,z),(g(u,v,w),g(v,u,w),g(w,v,u))),D((u,v,w),(u,v,w),...,(u,v,w),(f(x,y,z),f(y,x,z),f(z,y,x)))} *(3.2.1)*

for all $x,y,z,u,v,w\in X$ with $x\leq u$, $y\geq v$ and $z\leq w$ (iii) either f or g is continuous

Then f and g have a tripled common fixed point in X.

Proof.Let (x_0,y_0,z_0) be a given point in $X\times X\times X$,satisfying (i).

Write $x_1=f(x_0,y_0,z_0)$, $y_1=f(y_0,x_0,z_0)$, $z_1=f(z_0,y_0,x_0)$ *and* $x_2=g(x_1,y_1,z_1)$, $y_2=fg,x_1,z_1)$, $z_2=fg,y_1,x_1)$

Define the sequences $\{x_k\}$, $\{y_k\}$ and $\{z_k\}$inductively

$x_{k+1}=f(x_k,y_k,z_k)$, $y_{k+1}=f(y_k,x_k,z_k)$, $z_{k+1}=f(z_k,y_k,x_k)$

$x_{k+2}=g(x_{k+1},y_{k+1},z_{k+1})$, $y_{k+2}=g(y_{k+1},x_{k+1},z_{k+1})$, $z_{k+2}=g(z_{k+1},y_{k+1},x_{k+1})$

for all $k\in N$ *(3.2.2)*

Since $x_0\leq f(x_0,y_0,z_0)$,$y_0\geq f(y_0,x_0,z_0)$ *and* $z_0\leq f(z_0,y_0,x_0)$

and since (f,g) has mixed weakly monotone property, we get

$x_1=f(x_0,y_0,z_0)\leq g(f(x_0,y_0,z_0),f(y_0,x_0,z_0),f(z_0,y_0,x_0))=g(x_1,y_1,z_1)=x_2\Rightarrow x_1\leq x_2$,

$x_2=g(x_1,y_1,z_1)\leq f(g(x_1,y_1,z_1),g(y_1,x_1,z_1),fg,y_1,x_1))=f(x_2,y_2,z_2)=x_3\Rightarrow x_2\leq x_3$

and $y_1=f(y_0,x_0,z_0)\geq g(f(y_0,x_0,z_0),f(x_0,y_0,z_0),f(z_0,y_0,x_0))=g(y_1,x_1,z_1)=y_2\Rightarrow y_1\geq y_2$,

$y_2=f(y_1,x_1,z_1\geq f(g(y_1,x_1,z_1),g(x_1,y_1,z_1),g(z_1,y_1,x_1))=f(y_2,x_2,z_2=y_3\Rightarrow y_2\geq y_3$

also

$z_1 = f(z_0, y_0, x_0) \le g(f(z_0, y_0, x_0), f(y_0, x_0, z_0), f(x_0, y_0, z_0)) = g(z_1, y_1, x_1) = z_2 \Longrightarrow z_1 \le z_2,$

$$z_2 = f(z_1, y_1, x_1 \le f(g(z_1, y_1, x_1), g(y_1, x_1, z_1), g(x_1, y_1, z_1)) = f(z_2, y_2, x_2) = z_3 \Longrightarrow z_2 \le z_3$$

By induction,

$x_0 \le x_1 \le x_2 \le ......\le x_k \le x_{k+1} \le ....$

$y_0 \ge y_1 \ge y_2 \ge ......\ge y_k \ge y_{k+1} \ge ....$ *(3.2.3)*

$z_0 \le z_1 \le z_2 \le ......\le z_k \le z_{k+1} \le ....$ *for all* $k \in N$

Now show that these sequences are Cauchy

Define $D_k : X^3 \longrightarrow X$ *by*

$$D_k = D\begin{pmatrix} (x_k, y_k, z_{k)}, (x_k, y_k, z_{k)}, \dots (x_k, y_k, z_{k)}, (x_{k+1}, y_{k+1}, z_{k+1)}, \\ n-1\ times \end{pmatrix}$$

$$= A_b\begin{pmatrix} x_{k,,x_k}, \dots, x_k, x_{k+1} \\ n-1\ times \end{pmatrix} + A_b\begin{pmatrix} y_{k,,y_k}, \dots, y_k, y_{k+1} \\ n-1\ times \end{pmatrix} + A_b\begin{pmatrix} z_{k,,z_k}, \dots, z_k, z_{k+1} \\ n-1\ times \end{pmatrix}$$

for k=0,1,2....

$D_{2k+1} = A_b\ (x_{2k+1}, x_{2k+1}, \dots x_{2k+1}, x_{2k+2}) + A_b\ (y_{2k+1}, y_{2k+1}, \dots y_{2k+1}, y_{2k+2}) + A_b$
$(z_{2k+1}, z_{2k+1}, \dots z_{2k+1}, z_{2k+2})$

$= A_b(f(x_{2k}, y_{2k}, z_{2k}), f(x_{2k}, y_{2k}, z_{2k}) \dots f(x_{2k}, y_{2k}, z_{2k}), g(x_{2k+1}, y_{2k+1}, z_{2k+1})) + A_b$
$(f(y_{2k}, x_{2k}, z_{2k}), f(y_{2k}, x_{2k}, z_{2k}), \dots f(y_{2k}, x_{2k}, z_{2k}), g(y_{2k+1}, x_{2k+1}, z_{2k+1})) + A_b$
$(f(z_{2k}, y_{2k}, x_{2k}), f(z_{2k}, y_{2k}, x_{2k}), \dots f(z_{2k}, y_{2k}, x_{2k}), , g(z_{2k+1}, y_{2k+1}, x_{2k+1}))$

$\le \alpha\ max\ \{ D((x_{2k}, y_{2k}, z_{2k}), (x_{2k}, y_{2k}, z_{2k}), \dots (x_{2k}, y_{2k}, z_{2k}), (x_{2k+1}, y_{2k+1}, z_{2k+1})),$
$D((x_{2k}, y_{2k}, z_{2k}), (x_{2k}, y_{2k}, z_{2k}), , \dots (x_{2k}, y_{2k}, z_{2k}), (f(x_{2k}, y_{2k}, z_{2k}), f(y_{2k}, x_{2k}, z_{2k}), f(z_{2k}, y_{2k}, x_{2k}))),$
$D((x_{2k+1}, y_{2k+1}, z_{2k+1}), (x_{2k+1}, y_{2k+1}, z_{2k+1}), \dots (x_{2k+1}, y_{2k+1}, z_{2k+1}),$
$(g(x_{2k+1}, y_{2k+1}, z_{2k+1}), g(y_{2k+1}, x_{2k+1}, z_{2k+1}), g(z_{2k+1}, y_{2k+1}, x_{2k+1}))),$
$D((x_{2k}, y_{2k}, z_{2k}), (x_{2k}, y_{2k}, z_{2k}), , \dots (x_{2k}, y_{2k}, z_{2k}),$
$(g(x_{2k+1}, y_{2k+1}, z_{2k+1}), g(y_{2k+1}, x_{2k+1}, z_{2k+1}), g(z_{2k+1}, y_{2k+1}, x_{2k+1}))),$
$D((x_{2k+1}, y_{2k+1}, z_{2k+1}), (x_{2k+1}, y_{2k+1}, z_{2k+1}), \dots (x_{2k+1}, y_{2k+1}, z_{2k+1}), (f(x_{2k}, y_{2k}, z_{2k}), f(y_{2k}, x_{2k}, z_{2k}),$
$f(z_{2k}, y_{2k}, x_{2k})))\}$

From (3.2.2)

$D_{2k+1} \le \alpha\ max\ \{$ D((x_{2k}, y_{2k}, z_{2k}) , (x_{2k}, y_{2k}, z_{2k}) , $\dots (x_{2k}, y_{2k}, z_{2k}), (x_{2k+1}, y_{2k+1}, z_{2k+1})),$
D((x_{2k}, y_{2k}, z_{2k}) , (x_{2k}, y_{2k}, z_{2k}) ,,... (x_{2k}, y_{2k}, z_{2k}) , $,(x_{2k+1}, y_{2k+1}, z_{2k+1})),$
D($(x_{2k+1}, y_{2k+1}, z_{2k+1}), (x_{2k+1}, y_{2k+1}, z_{2k+1})$,... $(x_{2k+1}, y_{2k+1}, z_{2k+1})$, $(x_{2k+2}, y_{2k+2}, z_{2k+2})$),
D((x_{2k}, y_{2k}, z_{2k}) , (x_{2k}, y_{2k}, z_{2k}) , $\dots (x_{2k}, y_{2k}, z_{2k})$, $(x_{2k+2}, y_{2k+2}, z_{2k+2})$),
D($(x_{2k+1}, y_{2k+1}, z_{2k+1}), (x_{2k+1}, y_{2k+1}, z_{2k+1})$,... $(x_{2k+1}, y_{2k+1}, z_{2k+1})$, $(x_{2k+1}, y_{2k+1}, z_{2k+1})$) }
(3.2.4)

$= \alpha\ max\ \{$ D((x_{2k}, y_{2k}, z_{2k}) , (x_{2k}, y_{2k}, z_{2k}) , $\dots (x_{2k}, y_{2k}, z_{2k}), (x_{2k+1}, y_{2k+1}, z_{2k+1})),$
D((x_{2k}, y_{2k}, z_{2k}) , (x_{2k}, y_{2k}, z_{2k}) ,,... (x_{2k}, y_{2k}, z_{2k}) , $,(x_{2k+1}, y_{2k+1}, z_{2k+1})),$
D($(x_{2k+1}, y_{2k+1}, z_{2k+1}), (x_{2k+1}, y_{2k+1}, z_{2k+1})$,... $(x_{2k+1}, y_{2k+1}, z_{2k+1})$, $(x_{2k+2}, y_{2k+2}, z_{2k+2})$),
D($(x_{2k}, y_{2k}, z_{2k}), (x_{2k}, y_{2k}, z_{2k}), \dots (x_{2k}, y_{2k}, z_{2k}), (x_{2k+2}, y_{2k+2}, z_{2k+2}))\}$

From lemma 2.10,

$D_{2k+1} \le \alpha\ max\ \{$ D((x_{2k}, y_{2k}, z_{2k}) , (x_{2k}, y_{2k}, z_{2k}) , $\dots (x_{2k}, y_{2k}, z_{2k}), (x_{2k+1}, y_{2k+1}, z_{2k+1})),$
D((x_{2k}, y_{2k}, z_{2k}) , (x_{2k}, y_{2k}, z_{2k}) ,,... (x_{2k}, y_{2k}, z_{2k}) , $,(x_{2k+1}, y_{2k+1}, z_{2k+1})),$
D($(x_{2k+1}, y_{2k+1}, z_{2k+1}), (x_{2k+1}, y_{2k+1}, z_{2k+1}), \dots (x_{2k+1}, y_{2k+1}, z_{2k+1}), (x_{2k+2}, y_{2k+2}, z_{2k+2})), (n-1)b$

D((x_{2k}, y_{2k}, z_{2k}) , (x_{2k}, y_{2k}, z_{2k}) , $\ldots.(x_{2k}, y_{2k}, z_{2k}), (x_{2k+1}, y_{2k+1}, z_{2k+1})$) $+b^2$D($(x_{2k+1}, y_{2k+1}, z_{2k+1}), (x_{2k+1}, y_{2k+1}, z_{2k+1})$,...$(x_{2k+1}, y_{2k+1}, z_{2k+1}), (x_{2k+2}, y_{2k+2}, z_{2k+2})$)}

$\leq \alpha(n-1)b$ $D($ (x_{2k}, y_{2k}, z_{2k}) , (x_{2k}, y_{2k}, z_{2k}) , $\ldots.(x_{2k}, y_{2k}, z_{2k}), (x_{2k+1}, y_{2k+1}, z_{2k+1}))$ $+b^2D((x_{2k+1}, y_{2k+1}, z_{2k+1}), (x_{2k+1}, y_{2k+1}, z_{2k+1}),\ldots(x_{2k+1}, y_{2k+1}, z_{2k+1}), (x_{2k+2}, y_{2k+2}, z_{2k+2}))$

$= \alpha(n-1)b[A_b\ (x_{2k}, x_{2k}, \ldots. x_{2k}, x_{2k+1}) + A_b\ (y_{2k}, y_{2k}, \ldots.. y_{2k}, y_{2k+1}\ ) + A_b\ (z_{2k}, z_{2k}, \ldots.. z_{2k}, z_{2k+1})]$ $+\ b^2\ [\ A_b\ (\ x_{2k+1}\ ,\ x_{2k+1}\ ,\ldots.\ x_{2k+1}\ ,\ x_{2k+2}) + A_b\ (\ y_{2k+1}\ ,\ y_{2k+1}, \ldots.. y_{2k+1}, y_{2k+2}\ ) + A_b$ $(z_{2k+1}, z_{2k+1}, \ldots.. z_{2k+1}, z_{2k+2})]$ (3.2.5)

Therefore

$D_{2k+1} \leq \alpha(n-1)b\ [\ A_b\ (\ x_{2k}\ ,\ x_{2k}\ ,\ldots.\ x_{2k}\ ,\ x_{2k+1}) + A_b\ (\ y_{2k}\ ,\ y_{2k}, \ldots.. y_{2k}, y_{2k+1}\ ) + A_b$ $(z_{2k}, z_{2k}, \ldots.. z_{2k}, z_{2k+1})] + b^2\ [\ A_b\ (x_{2k+1}, x_{2k+1}, \ldots. x_{2k+1}, x_{2k+2}) + A_b\ (y_{2k+1}, y_{2k+1}, \ldots.. y_{2k+1}, y_{2k+2}$ $) + A_b\ (z_{2k+1}, z_{2k+1}, \ldots.. z_{2k+1}, z_{2k+2})]$ (3.2.6)

$$\Longrightarrow (1-\alpha\, b^2)D_{2k+1} \leq \alpha\,(n-1)bD_{2k}$$

$\Longrightarrow D_{2k+1} \leq \frac{\alpha\,(n-1)b}{(1-\alpha\, b^2)} D_{2k}$ (3.2.7)

Put $\beta = \frac{\alpha\,(n-1)b}{(1-\alpha\, b^2)}$, *then* $0 \leq \beta < 1$

From (3.2.7), $D_{2k+1} \leq \beta D_{2k}$

Similarly we can show that $D_{2k+2} \leq \beta D_{2k+1}$*for k=0,1,2,...*

Hence $D_{k+1} \leq \beta D_k$

Therefore

$D_{k+1} \leq \beta^{k+1} D_0$(3.2.8)

Define $D_{k,l} = D\begin{pmatrix} (x_k, y_k, z_k), (x_k, y_k, z_k) \ldots. (x_k, y_k, z_k), (x_l, y_l, z_l), \\ n-1\ times \end{pmatrix}$

$$= A_b\begin{pmatrix} x_k, x_k, \ldots.., x_k, x_l \\ n-1\ times \end{pmatrix} + A_b\begin{pmatrix} y_k, y_k, \ldots.., y_k, y_l \\ n-1\ times \end{pmatrix} + A_b\begin{pmatrix} z_k, z_k, \ldots.., z_k, z_l \\ n-1\ times \end{pmatrix}$$

Now we have to show that $D_{k,l}$ *is a Cauchy sequence*

By lemma 2.10, for all $k, m \in N, k \leq m$

we have

$$D_{k+1,m+1} = A_b\begin{pmatrix} x_{k+1}, x_{k+1}, \ldots.., x_{k+1}, x_{m+1} \\ n-1\ times \end{pmatrix} + A_b\begin{pmatrix} y_{k+1}, y_{k+1}, \ldots.., y_{k+1}, y_{m+1} \\ n-1\ times \end{pmatrix} + A_b\begin{pmatrix} z_{k+1}, z_{k+1}, \ldots.., z_{k+1}, z_{m+1} \\ n-1\ times \end{pmatrix}$$

$\leq (n-1)b\,[A_b\binom{x_{k+1,,}x_{k+1},\dots,x_{k+1},\,x_{k+2}}{n-1\ times\quad ⬚} + A_b\binom{y_{k+1,,}y_{k+1},\dots,y_{k+1},\,y_{k+2}}{n-1\ times\quad ⬚} + A_b\binom{z_{k+1,,}z_{k+1},\dots,z_{k+1},\,z_{k+2}}{n-1\ times\quad ⬚}]$

$+b^2\,[A_b\binom{x_{k+2,,}x_{k+2},\dots,x_{k+2},\,x_{m+1}}{n-1\ times\quad ⬚} + A_b\binom{y_{k+2,}y_{k+2},\dots,y_{k+2},\,y_{m+1}}{n-1\ times\quad ⬚} + A_b\binom{z_{k+2,,}z_{k+2},\dots,z_{k+2},\,z_{m+1}}{n-1\ times\quad ⬚}]$

$= (n-1)bD_{k+1} + b^2(n-1)b[\,A_b\binom{x_{k+2,,}x_{k+2},\dots,x_{k+2},\,x_{k+3}}{n-1\ times\quad ⬚} + A_b\binom{y_{k+2,}y_{k+2},\dots,y_{k+2},\,y_{k+3}}{n-1\ times\quad ⬚} +$
$A_b\binom{z_{k+2,,}z_{k+2},\dots,z_{k+2},\,z_{k+3}}{n-1\ times\quad ⬚}\,]$

$+b^2b^2\,[A_b\binom{x_{k+3,,}x_{k+3},\dots,x_{k+3},\,x_{m+1}}{n-1\ times\quad ⬚} + A_b\binom{y_{k+3,}y_{k+3},\dots,y_{k+3},\,y_{m+1}}{n-1\ times\quad ⬚} + A_b\binom{z_{k+3,,}z_{k+3},\dots,z_{k+3},\,z_{m+1}}{n-1\ times\quad ⬚}]$

$= (n-1)bD_{k+1} + b^3(n-1)D_{k+2} + b^5(n-1)D_{k+3} + b^{2(m-k)-3}(n-1)\ \ )\ \ [A_b\binom{x_{m-1,,}x_{m-1},\dots,x_{m-1},\,x_m}{n-1\ times\quad ⬚} +$
$A_b\binom{y_{m-1,}y_{m-1},\dots,y_{m-1},\,y_m}{n-1\ times\quad ⬚} + A_b\binom{z_{m-1,,}z_{m-1},\dots,z_{m-1},\,z_m}{n-1\ times\quad ⬚}] + b^{2(m-k)-1}(n-1)\ \ )\ \ [A_b\binom{x_{m,,}x_m,\dots,x_m,\,x_{m+1}}{n-1\ times\quad ⬚} +$
$A_b\binom{y_{m,}y_m,\dots,y_m,\,y_{m+1}}{n-1\ times\quad ⬚} + A_b\binom{z_{m,,}z_m,\dots,z_m,\,z_{m+1}}{n-1\ times\quad ⬚}$

From (3.2.8)

$D_{k+1,m+1} \leq (n-1)b[\beta^{k+1} + b^2\beta^{k+2} + b^4\beta^{k+3} + \cdots \ldots + b^{2(m-k)-2}\beta^m]\,D_0$

$$\Longrightarrow D_{k+1,m+1} \leq (n-1)b\beta^{k+1}[1 + b^2\beta + (b^2\beta)^2 + \cdots .. + (b^2\beta)^{(m-k-1)}]D_0$$

$$= (n-1)b\beta^{k+1}[1 + \gamma + \gamma^2 + \cdots .. + \gamma^{(m-k-1)}]D_0$$

$$\leq (n-1)b\beta^{k+1}\frac{1}{1-\gamma}\,D_0$$

Where $\gamma = b^2\beta$

Hence for all $k, m \in N, k \leq m$, *we have*

$$D_{k,m} = A_b\binom{x_{k,,}x_k,\ \dots\dots,x_k,x_m}{n-1\ times\quad ⬚} + A_b\binom{y_{k,,}y_k,\ \dots\dots,y_k,y_m}{n-1\ times\quad ⬚} + A_b\binom{z_{k,,}z_k,\ \dots\dots,z_k,z_m}{n-1\ times\quad ⬚}$$
$$\leq (n-1)b\beta^k\frac{1}{1-\gamma}\,D_0$$

Since $0 \leq \gamma = \alpha b^2((n-1)b+1) < 1$, *we have*

$$\lim_{k,m\to\infty} A_b\binom{x_{k,,}x_k,\ \dots\dots,x_k,x_m}{n-1\ times\quad ⬚} + A_b\binom{y_{k,,}y_k,\ \dots\dots,y_k,y_m}{n-1\ times\quad ⬚} + A_b\binom{z_{k,,}z_k,\ \dots\dots,z_k,z_m}{n-1\ times\quad ⬚} = 0$$

That is,
$$\lim_{k,m\to\infty} A_b\binom{x_{k,,}x_k,\ \dots\dots,x_k,x_m}{n-1\ times\quad ⬚} = \lim_{k,m\to\infty} A_b\binom{y_{k,,}y_k,\ \dots\dots,y_k,y_m}{n-1\ times\quad ⬚} = \lim_{k,m\to\infty} A_b\binom{z_{k,,}z_k,\ \dots\dots,z_k,z_m}{n-1\ times\quad ⬚} = 0$$

Therefore $\{x_k\},\{y_k\}$ *and* $\{z_k\}$*are Cauchy sequences in X.*

By the completeness of X, there exists $x, y, z \in X$ *such that* $x_k \to x$, $y_k \to y$ *and* $z_k \to z$ *as* $k \to \infty$

Therefore $D_{k,l}$ is a Cauchy sequence.

Now we show that (x, y, z) is a tripled fixed point of f and g

Without loss of generality, we may suppose that f is continuous, we have $x = \lim_{k\to\infty} x_{2k+1} = \lim_{k\to\infty} f(x_{2k}, y_{2k}, z_{2k}) = f\left(\lim_{k\to\infty} x_{2k}, \lim_{k\to\infty} y_{2k}, \lim_{k\to\infty} z_{2k}\right) = f(x, y, z)$

$$y = \lim_{k\to\infty} y_{2k+1} = \lim_{k\to\infty} f(y_{2k}, x_{2k}, z_{2k}) = f\left(\lim_{k\to\infty} y_{2k}, \lim_{k\to\infty} x_{2k}, \lim_{k\to\infty} z_{2k}\right) = f(y, x, z)$$

and

$$z = \lim_{k\to\infty} z_{2k+1} = \lim_{k\to\infty} f(z_{2k}, y_{2k}, x_{2k}) = f\left(\lim_{k\to\infty} z_{2k}, \lim_{k\to\infty} y_{2k}, \lim_{k\to\infty} x_{2k}\right) = f(z, y, x)$$

Thus (x, y, z) is a tripled fixed point of f.

From (3.2.1), we have

A_b *(x,x,....x,f(x,y,z))+* A_b*(y,y,....y,f(y,x,z))+* A_b*(z,z,....z,f(z,y,x))*

= A_b *(f(x,y,z),f(x,y,z),....f(x,y,z),g(x,y,z))+* A_b *(f(y,x,z),f(y,x,z),....f(y,x,z),g(y,x,z))+* A_b *(f(z,y,z),f(z,y,x),....f(z,y,x),g(z,y,x))*

$\leq$ α *max{ D((x,y,z),(x,y,z),...(x,y,z),(x,y,z)),D((x,y,z),(x,y,z),...(x,y,z),(f(x,y,z),f(y,x,z),f(z,y,x))),D((x,y,z),(x,y,z),...(x,y,z) ,(g(x,y,z),g(y,x,z),g(z,y,x))),D((x,y,z),(x,y,z),...(x,y,z),(g(x,y,z),g(y,x,z),g(z,y,x))),D((x,y,z),(x,y,z),...(x,y,z),(f(x,y,z),f(y,x,z),f(z,y,x)))}*

= α *max { 0, D((x,y,z),(x,y,z),...(x,y,z), (x,y,z)), D((x,y,z),(x,y,z),...(x,y,z),(g(x,y,z),g(y,x,z),g(z,y,x))),D((x,y,z),(x,y,z),...(x,y,z),(g(x,y,z),g(y,x,z),g(z,y,x))) ,D((x,y,z),(x,y,z),...(x,y,z), (x,y,z)),}*

= α *{ D((x,y,z),(x,y,z),...(x,y,z),(g(x,y,z),g(y,x,z),g(z,y,x)))}*

$\leq$ αb *((g(x,y,z),g(y,x,z),g(z,y,x)),(g(x,y,z),g(y,x,z),g(z,y,x)),...(g(x,y,z),g(y,x,z),g(z,y,x)),(x,y,z))*

Since $\alpha b < 1$ *, we have (g(x,y,z),g(y,x,z),g(z,y,x))=(x,y,z)*

$\Longrightarrow$ *g(x,y,z)=x, g(y,x,z)=y and g(z,y,x)=z*

Therefore (x,y,z) is a tripled fixed point of g.

Thus (x, y, z) is a tripled common fixed point of f *and* g.

Theorem 3.3.Let $(X, \leq, A)$ *be a partially ordered complete* A_b *−metric space and* $f, g: X^3 \to X$ *such that (i) the pair* (f, g) *has mixed weakly monotone property on* X *and there exist* x_0, y_0, z_0 *in* X *such that*

$x_0 \leq f(x_0, y_0, z_0)$, $f(y_0, x_0, z_0) \leq y_0$, *and* $z_0 \leq f(z_0, y_0, x_0)$ *or* $x_0 \leq g(x_0, y_0, z_0)$, $g(y_0, x_0, z_0) \leq y_0$, *and* $z_0 \leq g(z_0, y_0, x_0)$

(ii) there is an α *such that* $\alpha b^2((n-1)b+1) < 1$ *and*

A(f(x,y,z),f(x,y,z),....f(x,y,z),g(u,v,w))+A(f(y,x,z),f(y,x,z),....f(y,x,z),g(v,u,w)
+A(f(z,y,x),f(z,y,x),....f(z,y,x),g(w,v,u))

$\leq \alpha$ *max{D((x,y,z),(x,y,z),...(x,y,z),(u,v,w)), D((x,y,z),(x,y,z),...(x,y,z),(f(x,y,z),f(y,x,z),f(z,y,x))), D((u,v,w),(u,v,w),...,(u,v,w),(g(u,v,w),g(v,u,w)),g(w,v,u)),D((x,y,z),(x,y,z),...(x,y,z),(g(u,v,w),g(v,u,w),g(w,v, u))),D((u,v,w),(u,v,w),...,(u,v,w),(f(x,y,z),f(y,x,z),f(z,y,x)))}* *(3.2.1)*

for all $x, y, z, u, v, w \in X$ *with* $x \leq u$, $y \geq v$ *and* $z \leq w$

(iii) X has the following properties.

(a) if $\{x_k\}$ *is an increasing sequence with*

$x_k \to x$ *then* $x_k \leq x$ *for all* $k \in N$

(b) if $\{y_k\}$ *is a decreasing sequence with* $y_k \to y$ *, then* $y \leq y_k$ *for all* $k \in N$

(c) if $\{z_k\}$ *is an increasing sequence with*

$z_k \to z$ *then* $z_k \leq z$ *for all* $k \in N$

Then f and g have tripled common fixed point in X.

Proof.*Suppose X satisfies (a) and (b), by (3.2.3) we get* $x_k \leq x$, $y_k \geq y$ *and* $z_k \leq z$ *for all* $k \in N$

Applying lemmas 2.10 and 2.11, we have

D((x,y,z),(x,y,z),...(x,y,z),(f(x,y,z),f(y,x,z),f(z,y,x)))

$\leq$ b(n-1)D((x,y,z),(x,y,z),...(x,y,z), $(x_{2k+2}, y_{2k+2}, z_{2k+2})$) + b^2 D($(x_{2k+2}, y_{2k+2}, z_{2k+2})$, $(x_{2k+2}, y_{2k+2}, z_{2k+2})$,... $(x_{2k+2}, y_{2k+2}, z_{2k+2})$,(f(x,y,z),f(y,x,z),f(z,y,x)))

=b(n-1)D((x,y,z),(x,y,z),...(x,y,z),$(x_{2k+2}, y_{2k+2}, z_{2k+2})$)+$b^2$D((g$(x_{2k+1}, y_{2k+1}, z_{2k+1})$,g$(y_{2k+1}, x_{2k+1}, z_{2k+1})$, g $(z_{2k+1}, y_{2k+1}, x_{2k+1})$),(g $(x_{2k+1}, y_{2k+1}, z_{2k+1})$,g $(y_{2k+1}, x_{2k+1}, z_{2k+1})$,g $(z_{2k+1}, y_{2k+1}, x_{2k+1})$),...(g $(x_{2k+1}, y_{2k+1}, z_{2k+1})$,g$(y_{2k+1}, x_{2k+1}, z_{2k+1})$,g$(z_{2k+1}, y_{2k+1}, x_{2k+1})$),f(x,y,z),f(y,x,z),f(z,y,x)))

$\Longrightarrow$ D((x,y,z),(x,y,z),...(x,y,z),(f(x,y,z),f(y,x,z),f(z,y,x))) $\leq$ *b(n-1)* [A_b (x,x,...x, x_{2k+2})+ A_b (y,y,...y, y_{2k+2})+ A_b (z,z,...z, z_{2k+2})]+ b^2 A_b [*g* $(x_{2k+1}, y_{2k+1}, z_{2k+1})$,*g* $(x_{2k+1}, y_{2k+1}, z_{2k+1})$,...*g* $(x_{2k+1}, y_{2k+1}, z_{2k+1})$,f(x,y,z)]+ b^2 A_b [*g* $(y_{2k+1}, x_{2k+1}, z_{2k+1})$,*g* $(y_{2k+1}, x_{2k+1}, z_{2k+1})$,...*g* $(y_{2k+1}, x_{2k+1}, z_{2k+1})$,f(y,x,z)]+ b^2 A_b [*g* $(z_{2k+1}, y_{2k+1}, x_{2k+1})$,*g* $(z_{2k+1}, y_{2k+1}, x_{2k+1})$,...*g*$(z_{2k+1}, y_{2k+1}, x_{2k+1})$,f(z,y,x)]

By (3.2.1), we get

A_b [g $(x_{2k+1}, y_{2k+1}, z_{2k+1})$,g $(x_{2k+1}, y_{2k+1}, z_{2k+1})$,... g $(x_{2k+1}, y_{2k+1}, z_{2k+1})$,f(x,y,z)]+ A_b [g $(y_{2k+1}, x_{2k+1}, z_{2k+1})$, g $(y_{2k+1}, x_{2k+1}, z_{2k+1})$,... g $(y_{2k+1}, x_{2k+1}, z_{2k+1})$,f(y,x,z)]+ A_b [$g(z_{2k+1}, y_{2k+1}, x_{2k+1}), g(z_{2k+1}, y_{2k+1}, x_{2k+1})$,... $g(z_{2k+1}, y_{2k+1}, x_{2k+1})$,f(z,y,x)]

$\leq \alpha$ max { D($(x_{2k+1}, y_{2k+1}, z_{2k+1})$, $(x_{2k+1}, y_{2k+1}, z_{2k+1})$, $\ldots.(x_{2k+1}, y_{2k+1}, z_{2k+1}), (x, y, z))$, D($(x_{2k+1}, y_{2k+1}, z_{2k+1})$, $(x_{2k+1}, y_{2k+1}, z_{2k+1})$, $\ldots.(x_{2k+1}, y_{2k+1}, z_{2k+1})$, $(g(x_{2k+1}, y_{2k+1}, z_{2k+1}), g(y_{2k+1}, x_{2k+1}, z_{2k+1})$,

$g(z_{2k+1}, y_{2k+1}, x_{2k+1}))$, D((x,y,z),(x,y,z),...(x,y,z),(f(x,y,z),f(y,x,z),f(z,y,x))), D($(x_{2k+1}, y_{2k+1}, z_{2k+1})$, $(x_{2k+1}, y_{2k+1}, z_{2k+1})$, $\ldots.(x_{2k+1}, y_{2k+1}, z_{2k+1}), (f(x, y, z), f(y, x, z), f(z, y, x)))$,

D((x,y,z),(x,y,z),...(x,y,z),$(g(x_{2k+1}, y_{2k+1}, z_{2k+1}), g(y_{2k+1}, x_{2k+1}, z_{2k+1})$,

$g(z_{2k+1}, y_{2k+1}, x_{2k+1}))\}$

= α max { D($(x_{2k+1}, y_{2k+1}, z_{2k+1})$, $(x_{2k+1}, y_{2k+1}, z_{2k+1})$, $\ldots.(x_{2k+1}, y_{2k+1}, z_{2k+1}), (x, y, z))$, D($(x_{2k+1}, y_{2k+1}, z_{2k+1})$,$(x_{2k+1}, y_{2k+1}, z_{2k+1})$,$\ldots.(x_{2k+1}, y_{2k+1}, z_{2k+1}), (x_{2k+2}, y_{2k+2}, z_{2k+2}))$,

D((x,y,z),(x,y,z),...(x,y,z),(f(x,y,z),f(y,x,z),f(z,y,x))),D($(x_{2k+1}, y_{2k+1}, z_{2k+1})$, $(x_{2k+1}, y_{2k+1}, z_{2k+1})$, $\ldots.(x_{2k+1}, y_{2k+1}, z_{2k+1}), (f(x, y, z), f(y, x, z), f(z, y, x)))$, D((x,y,z),(x,y,z),...(x,y,z), $(x_{2k+2}, y_{2k+2}, z_{2k+2}))\}$

From (3.2.6) and (3.2.7)

D((x,y,z),(x,y,z),...(x,y,z),(f(x,y,z),f(y,x,z),f(z,y,x))) $\leq$ *b(n-1)[* A_b *(x,x,...x,* x_{2k+2} *)+* A_b *(y,y,...y,* y_{2k+2} *)+* A_b *(z,z,...z,* z_{2k+2} *)]+* $b^2\alpha$ *max {* D($(x_{2k+1}, y_{2k+1}, z_{2k+1})$,$(x_{2k+1}, y_{2k+1}, z_{2k+1})$,$\ldots.(x_{2k+1}, y_{2k+1}, z_{2k+1}), (x, y, z))$,

D($(x_{2k+1}, y_{2k+1}, z_{2k+1})$, $(x_{2k+1}, y_{2k+1}, z_{2k+1})$, $\ldots.(x_{2k+1}, y_{2k+1}, z_{2k+1}), (x_{2k+2}, y_{2k+2}, z_{2k+2}))$ D((x,y,z),(x,y,z),...(x,y,z),(f(x,y,z),f(y,x,z),f(z,y,x))),D($(x_{2k+1}, y_{2k+1}, z_{2k+1})$, $(x_{2k+1}, y_{2k+1}, z_{2k+1})$, $\ldots.(x_{2k+1}, y_{2k+1}, z_{2k+1}), (f(x, y, z), f(y, x, z), f(z, y, x)))$,
D((x,y,z),(x,y,z),...(x,y,z),$(x_{2k+2}, y_{2k+2}, z_{2k+2}))\}$

Taking the limit as $k \longrightarrow \infty$ in (3.2.1), we obtain

D((x,y,z),(x,y,z),...(x,y,z),(f(x,y,z),f(y,x,z),f(z,y,x))) $\leq$ *b(n-1) [* A_b *(x,x,...x,x)+* A_b *(y,y,...y,y)+* A_b *(z,z,...z,z)] +*$b^2\alpha$ *max { D((x,y,z),(x,y,z),...(x,y,z),(x,y,z)),*

D((x,y,z),(x,y,z),...(x,y,z),(x,y,z)), D((x,y,z),(x,y,z),...(x,y,z),(f(x,y,z),f(y,x,z),f(z,y,x))), D((x,y,z),(x,y,z),...(x,y,z),(f(x,y,z),f(y,x,z),f(z,y,x))), D((x,y,z),(x,y,z),...(x,y,z),(x,y,z))}

=$b^2\alpha$ *D((x,y,z),(x,y,z),...(x,y,z),(f(x,y,z),f(y,x,z),f(z,y,x)))*

Since $b^2\alpha$*<1, we have*

D((x,y,z),(x,y,z),...(x,y,z),(f(x,y,z),f(y,x,z),f(z,y,x)))=0

$\Longrightarrow$ *(f(x,y,z),f(y,x,z),f(z,y,x))=(x,y,z)*

That is, f(x,y,z)=x , f(y,x,z)=y and f(z,y,x)=z

Therefore (x,y,z) is a tripled fixed point of f.

Similarly we can show that g(x,y,z)=x , g(y,x,z)=y and g(z,y,x)=z

Hence f(x,y,z)=x=g(x,y,z) , f(y,x,z)=y=g(y,x,z) and f(z,y,x)=z=g(z,y,x)

Thus (x, y, z) is a tripled common fixed point of f and g.

Theorem 3.4.*Suppose Theorem 3.2 or Theorem 3.3 satisfied,*

if further $\{x_n\}$ *is an increasing sequence with*

$x_n \to x$ *and* $x_n \leq u$ *for each n, then* $x \leq u$ *. Then f and g have a unique tripled common fixed points.*

Further more, any fixed point of f is a fixed point of g, and conversely.

Proof.*Suppose the given condition holds,*

Let (x,y,z) and (u,v,w) in $X \times X \times X$*, there exist* (x^*, y^*, z^*) *in* $X \times X \times X$ *, that is, comparable to (x,y,z) and (u,v,w)*

D((x,y,z),(x,y,z),...(x,y,z),(u,v,w))= A_b*(x,x,...x,u)+* A_b*(y,y,...y,u)+* A_b*(z,z,...z,w)*

= A_b *(f(x,y,z),f(x,y,z),...f(x,y,z),g(u,v,w))+* A_b*(f(y,x,z),f(y,x,z),...f(y,x,z),g(v,u,w))+* A_b*(f(z,y,x),f(z,y,x),...f(z,y,x),g(w,v,u))*

$\leq \alpha$ *max { D((x,y,z),(x,y,z),...(x,y,z),(u,v,w)),D((x,y,z),(x,y,z),...(x,y,z),(f(x,y,z),f(y,x,z),f(z,y,x))),D((u,v,w),(u,v,w),...(u,v,w),(g(u,v,w),g(v,u,w),g(w,v,u))), D((x,y,z),(x,y,z),...(x,y,z),(g(u,v,w),g(v,u,w),g(w,v,u))), D((u,v,w),(u,v,w),...(u,v,w),(f(x,y,z),f(y,x,z),f(z,y,x)))}*

= α *max { D((x,y,z),(x,y,z),...(x,y,z),(u,v,w)),D((x,y,z),(x,y,z),...(x,y,z),(x,y,z)), D((u,v,w),(u,v,w),...(u,v,w),(u,v,w)), D((x,y,z),(x,y,z),...(x,y,z),(u,v,w)), D((u,v,w),(u,v,w),...(u,v,w),(x,y,z))}*

= α *max { D((x,y,z),(x,y,z),...(x,y,z),(u,v,w)), D((x,y,z),(x,y,z),...(x,y,z),(u,v,w)), D((u,v,w),(u,v,w),...(u,v,w),(x,y,z))}*

$\leq \alpha$ *max { D((x,y,z),(x,y,z),...(x,y,z),(u,v,w)),*

D((x,y,z),(x,y,z),...(x,y,z),(u,v,w)),

b D((x,y,z),(x,y,z),...(x,y,z),(u,v,w))}

= α *b D((x,y,z),(x,y,z),...(x,y,z),(u,v,w))*

Since α *b <1, so that*

D((x,y,z),(x,y,z),...(x,y,z),(u,v,w))=0

$\Longrightarrow$*(x,y,z)=(u,v,w)* $\Longrightarrow$*x=u, y=v and z=w*

Suppose (x,y,z) and (x^*, y^*, z^*) *are tripled common fixed points such that* $x \le x^*$, $y \ge y^*$ *and* $z \le z^*$*, then* $x=x^*$, $y=y^*$ *and* $z=z^*$.

Now $D((x,y,z),(x,y,z),...(x,y,z),(x^*, y^*,z^*)) = A_b\ (x,x,...x,x^*)+ A_b(y,y,...y,y^*)+ A_b(z,z,...z,z^*)$

$= A_b\ (f(x,y,z),f(x,y,z),...f(x,y,z),g(x^*,y^*,z^*))+ A_b\ (f(y,x,z),f(y,x,z),... f(y,x,z),g(y^*,x^*,z^*))+ A_b\ (f(z,y,x),f(z,y,x),...f(z,y,x),g(z^*,y^*,x^*))$

$\le \alpha \max \{ D((x,y,z),(x,y,z),...(x,y,z),(x^*,y^*,z^*)), D((x,y,z),(x,y,z),...(x,y,z),(f(x,y,z),f(y,x,z),f(z,y,x))), D((x^*,y^*,z^*),(x^*,y^*,z^*),...(x^*,y^*,z^*),(g(x^*,y^*,z^*),g(y^*,x^*,z^*),g(z^*,y^*,x^*))),D((x,y,z),(x,y,z),...(x,y,z),(g(x^*,y^*,z^*),g(y^*,x^*,z^*),g(z^*,y^*,x^*))),D((x^*,y^*,z^*),(x^*,y^*,z^*),...(x^*,y^*,z^*),(f(x,y,z),f(y,x,z),f(z,yx)))\}$

$= \alpha \max \{ D((x,y,z),(x,y,z),...(x,y,z),(x,y,z)), D((x,y,z),(x,y,z),...(x,y,z),(x,y,z)), D((x,y,z),(x,y,z),...(x,y,z),(x^*,y^*,z^*)), D((x,y,z),(x,y,z),...(x,y,z),(x^*,y^*,z^*)), D((x,y,z),(x,y,z),...(x,y,z),(x,y,z))\}$

$\le \alpha b\ D((x^*,y^*,z^*),(x^*,y^*,z^*),...(x^*,y^*,z^*),(x,y,z))$

Since $\alpha\ b <1$*, so that*

$D((x^*,y^*,z^*),(x^*,y^*,z^*),...(x^*,y^*,z^*),(x,y,z))=0$

$\Longrightarrow (x^*,y^*,z^*)=(x,y,z)$

$\Longrightarrow x=x^* ,y=y^*$ *and* $z=z^*$

we show that any fixed point of f *is a fixed point of* g*, and conversely.*

That is, to show that (x,y,z) is a fixed point of f $\Longleftrightarrow$*(x,y,z) is a fixed point of* g.

Suppose that (x,y,z) is a tripled fixed point of f.

$D((x,y,z),(x,y,z),...(x,y,z),(g(x,y,z),g(y,x,z),g(z,y,x)))$

$= A_b(f(x,y,z),f(x,y,z),...f(x,y,z),g(x,y,z))+ A_b(f(y,x,z),f(y,x,z),...f(y,x,z),g(y,x,z))+ A_b(f(z,y,x),f(z,y,x),...f(z,y,x),g(z,y,x))$

$\le \alpha \max \{ D((x,y,z),(x,y,z),...(x,y,z),(x,y,z)), D((x,y,z),(x,y,z),...(x,y,z),(f(x,y,z),f(y,x,z),f(z,y,x))),D((x,y,z),(x,y,z),...(x,y,z),(g(x,y,z),g(y,x,z),g(z,y,x))),D((x,y,x),(x,y,z),...(x,y,z),(g(x,y,z),g(y,x,z),g(z,y,x))),D((x,y,z),(x,y,z),...(x,y,z),(f(x,y,z),f(y,x,z),f(z,y,x)))\}$

$= \alpha\ D((x,y,z),(x,y,z),...(x,y,z),(g(x,y,z),g(y,x,z),g(z,y,x))$

$\le \alpha\ b\ D((g(x,y,z),g(y,x,z),g(z,y,x)),(g(x,y,z),g(y,x,z),g(z,y,x)),..... (g(x,y,z),g(y,x,z),g(z,y,x)),(x,y,z))$

Since α b <1, we have

$$D((g(x,y,z),g(y,x,z),g(z,y,x)),(g(x,y,z),g(y,x,z),g(z,y,x)),.....(g(x,y,z),g(y,x,z),g(z,y,x)),(x,y,z))=0$$

$$\Rightarrow (g(x,y,z),g(y,x,z),g(z,y,x))=(x,y,z)$$

$$\Rightarrow x=g(x,y,z),\ y=g(y,x,z) \text{ and } z=g(z,y,x)$$

Therefore (x, y, z) is a tripled fixed point of g, and conversely.

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