https://doi.org/10.48047/AFJBS.6.13.2024.7474-7494

AFJBS

African Journal of Biological Sciences

Journal homepage: http://www.afjbs.com

Research Paper Open Access

A Two-Phase Heterogeneous Service Facility Queue with Encouraged Queue Length Distribution, Failure, And Vacation

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Volume 6, Issue 13, Aug 2024

Received: 15 June 2024

Accepted: 25 July 2024

Published: 15 Aug 2024

doi: 10.48047/AFJBS.6.13.2024.7474-7494

Abstract
We examine a two-phase heterogeneous service facility queue in this research, considering server vacation, failure, and the recommended encouraged queue length distribution. We regard client reneging as a result of server vacation in our queuing system. Following this, we determined the steady-state solutions and examined the expected length of the encouraged queue and the expected waiting period for clients by using the supplementary variable approach. Using numerical examples, we presented the effect of encouraged client arrival on queue length and cost analysis.

Keywords: Encouraged queue length distribution; Supplementary variable method; Reneging; server vacation.

MSC2020-Mathematics Subject Classification: 90B22, 60K30.

1. Introduction

A queuing theory addresses scenarios where certain items are being serviced while others are waiting for their turn. Queues form when the number of items needing service exceeds the capacity of the service facilities to process them. Every day, people wait in lines at various locations, such as banks, clinics, and food courts. In the literature on queuing theory, there is an increased focus and significance on the study of queuing systems with server vacation. The server goes on vacation to make use of its idle period. Queuing systems with server vacations can simulate real-world queuing scenarios in several systems, including manufacturing, communication, and production inventories.

The significance of the Supplementary Variable Technique (SVT) in stochastic models with remaining or delayed periods as the supplementary variable was studied in [1]. [2], have studied the approaches in various kinds of queuing problems, such as reneging and balking. A study has examined a single-server queuing system characterized by a general bulk service, non-Markovian group arrivals, multiple vacations, a standby server, balking behavior, varying arrival rates, and the option for a second server to perform repairs if the primary server fails in [3]. A concept of bulk service queuing model that has a single primary server, a backup server, two stages of heterogeneous service, and the ability to start failing under several conditions is studied in [4]. Many researchers have shown interest in studying client impatience, and several have made significant contributions to this field. One of the earliest studies on reneging and balking was done by [13, 14]. The initial study on reneging also investigated service rates and deterministic reneging with a single server Markovian arrival, as detailed in [5]. As examined in [6], the steady-state solutions were obtained through the SVT, which was employed to calculate the expected queue size and expected waiting period. Using two service stages, the Bernoulli vacation schedule, and failure, it investigated the behavior of a bulk arrival retrial queue model in [7]. [8] described an objective to minimize costs by developing an expected cost function and formulating an optimization problem, utilizing the direct search method to determine optimal service rates during both failure periods and busier periods. [9] analyzed the steady-state behavior of group arrival queues with two phases of heterogeneous service. The Bernoulli schedules vacations under multiple vacation policies, allowing the server to continue taking vacations until a new group of clients arrives after two successive service phases or the first vacation. Numerous studies have explored this field, including [10], which investigated balking and reneging in vacation queuing models. [11] studied a queuing system with two services, deriving transforms for the queue length at departure epochs, the steady-state queue length, and the sojourn period of any arbitrary client. The first service involves group processing, while the second service provides individual attention to each member of the group. A multiple vacation queuing model, in which the service station was subject to failure during operation, is described in [12]. The utility and efficacy of the Markovian single-channel Bernoulli two-vacation symmetric queue with stochastic Markov-renewal process and promoted stationary queue size analysis were examined in [15]. A brief summary of recent vacation queuing system research is described in [16]. The dynamic behavior of the system and the relaxation period are determined for several case studies of interest to authorize steady-state approximations, as explained in [17]. [18] have studied the encouraged arrival line with feedback, balking, and maintaining reneged clients with quality control policies for the Markovian model. An analysis of a group arrival queuing system where the server may have periodic random failures can be found in [19]. [20] describes the way supplementary optional service, failures, and numerous vacations produced in a group influence encouraged arrival in the Markovian queuing model. [21] analyzed a single-server queuing-inventory system controlled by a group Markovian arrival process, following which group sizes were used to produce a finite first-order Markov chain. [22] developed an overview of bulk arrival and group service queues with vacations in queuing models. By using supplementary variables, we can implement a bulk arrival queue system on a single server with group size-based services, and working vacations [23]. As studied in [24], the single server queue with a distribution of service periods, failures, and vacations allows the server to decide to depart the system or continue serving clients after service completion. A multi-server queuing model incorporating balking and reneging is explored in [25]. In [26], a single-server bulk arrival queuing system with two stages of service is discussed. [27] examines a limited-capacity single-server Markovian feedback queuing model that includes reneging, balking, and client retention after reneging. Fundamental queuing theory concepts can be found in [28]. [29] describes a single-server Markovian queuing system with encouraged arrivals as well as a financial analysis of the business. [30] describes the concept of a working vacation queue using the $M/M/1$ variant with server failures and repairs. An $M/M/1/N$ queuing system, incorporating reneging, balking, server vacations, and an optimal service rate, is discussed in [31].

In this study, we describe a service-based queue with encouraged arrivals that occurs in two stages simultaneously. This model was developed by including many assumptions, such as reneging during system failure and vacation. Impatient clients may renege or leave the service after entering, especially during system failures or vacation periods. This is a reasonable assumption in certain ways, and in real life, queues are often used in certain circumstances.

This work proposes minimizing system size and operating cost of the cost analysis of a two-phase heterogeneous service facility queue with encouraged queue length distribution, failure, and vacation. An introduction is described in section 1. The two-phase service with encouraged queue length distribution for the mathematical model is described in section 2. Notations are provided in section 3. The system of governing equations is derived in section 4. Random period queue length distribution and performance measures are described in sections 5 and 6. Numerical illustrations are provided in section 7. Comparison tables are provided in section 8. To optimize the system's operating cost, the cost analysis is explained with an example in section 9. Results and discussion are given in section 10. Section 11 contains the conclusion.

2. The Mathematical Model

We are analyzing a queuing system where arrivals occur in groups. The system features a two-phase heterogeneous service facility with a queue length distribution, failures, and vacations. The server attends to one client at a period, following a 'first come, first served' approach. Using the following assumptions, we can describe the general distribution model of the queuing system:

- In a stochastic arrival process, clients arrive in groups of changing lengths. Let $\Omega o_i \, \mathrm{d}t (i = 1,2,3, \dots)$ represent the probability that i clients arrive during the period interval $(t, t + \mathrm{d}t]$, where $0 \le o_i \le 1$. The sum of all probabilities over all group lengths is equal to 1, and $\Omega > 0$ denotes the expected group arrival rate.
- The service period, modeled by the general distribution $Z(s)$ with density function $z(s)$, corresponds to the probability density function $\zeta(y_0)\mathrm{d}y$ for finishing service within the period interval $(y_0, y_0 + \mathrm{d}y_0\ ]$, where y_0 represents the delayed period, such that:

$$\zeta(y_0) = \frac{z(y_0)}{1 - Z(y_0)}$$

Therefore:

$$z(s) = \zeta(s)e^{-\int_0^s \zeta(y_0)\mathrm{d}y_0}$$

- After completing service, the server can either take a vacation with probability p or remain available to provide service with a probability of $1 - p$, where $0 \leq p \leq 1$.
- The duration of vacations follows an exponential distribution with rate $\omega > 0$, resulting in an expected vacation period of $1/\omega$.
- The system experiences random failures, following a Poisson process with an expected failure rate of $\kappa > 0$. When a failure occurs, the interrupted client is placed back at the front of the queue.
- After a failure, the system promptly initiates the revamp process. Let $\Phi(r)$ represent the general distribution function of revamp periods, and $\varphi(r)$ denote the corresponding density function. The quantity $\eta(y_0)\mathrm{d}y_0$ represents the conditional probability that the revamp is finishing within the period interval $(y_0, y_0 + \mathrm{d}y_0]$, where y_0 is the delayed revamp period, such that:

$$‘\eta(y_0) = \frac{\varphi(y_0)}{1 - \Phi(y_0)}$$

Therefore:

$$\varphi(r) = \eta(r)e^{-\int_0^r \eta(y_0)\mathrm{d}y_0}$$

3. Notations

We establish the following probabilities to describe various system conditions.

- $P_l(y_0, t)$ represents the probability of that period t, the server is actively determined by service to l clients (where $l \geq 0$) who are waiting in line, including the one currently being served. The delayed service period for this client is denoted by y_0. Consequently, $P_l(t) = \int_0^\infty P_l(y_0, t)\mathrm{d}y_0$ represents the probability of that period t, there are l clients in the line, all expecting service, regardless of the specific value of y_0.
- $H_l(y_0, t)$ denotes the probability of that period t, the system is under revamp with a delayed revamp period of y_0, and there are l clients (where $l \geq 0$) waiting in the queue for service. Thus, $H_l(t) = \int_0^\infty H_l(y_0, t)\ \mathrm{d}\ y_0$ represents the probability of that period t, there are l clients in the queue while the system is under revamp, regardless of the value of y_0.

- $M_l(t)$ is the probability of that period t, the server is on vacation and there are l clients (where $l \geq 0$) waiting in the queue for service.
- $Q(t)$ denotes the probability that at a period t, there are no clients in the system and the server is free and available for service.

$$P_j(y_0, g) = \sum_{l=1}^{\infty} = g^l P_{l,j}(y_0); P_j(g) = \sum_{l=1}^{\infty} g^l P_{l,j} |g| \leq 1; j = 1,2 \tag{3.1}$$

$$H(y_0, g) = \sum_{l=1}^{\infty} g^l R_l(y_0); H(g) = \sum_{l=1}^{\infty} g^l H_l |g| \leq 1 \tag{3.2}$$

$$M(y_0, g) = \sum_{l=1}^{\infty} g^l M_l(y_0); M(g) = \sum_{l=1}^{\infty} g^l M_l |g| \leq 1 \tag{3.3}$$

$$O(g) = \sum_{i=1}^{l} g^i o_i \tag{3.4}$$

4. The set differential difference equation of the model

Assuming the system reaches a stable equilibrium, we can analyze its various states by considering the limits as the period t approaches infinity (i.e., $t \to \infty$). For different system states, we define the following steady-state probabilities, $\lim_{t\to\infty} P_l(y_0, t) = P_l(y_0), \lim_{t\to\infty} P_l(t) =$ $\lim_{t\to\infty} \int_0^{\infty} P_l(y_0, t) \mathrm{d}y = P_l, \lim_{t\to\infty} H_l(y_0, t) = H_l(y_0), \lim_{t\to\infty} H_l(t) =$ $\lim_{t\to\infty} \int_0^{\infty} H_l(y_0, t) \mathrm{d}y_0 = \; H_l, \lim_{t\to\infty} M_l(t) = M_l, \lim_{t\to\infty} Q(t) = Q$ By linking the system's states at period $t + \mathrm{d}t$ to those t and then taking the limit as t approaches infinity, we derive the following equations describing the system's steady-state behavior.

$$\frac{\mathrm{d}}{\mathrm{d}y_0}P_{l,1}(y_0)+\{((\Upsilon+1)\Omega)+\zeta_1(y_0)+\kappa\}P_{l,2}(y_0)=((\Upsilon+1)\Omega)\sum_{i=1}^{l} o_iP_{l-i,1}(y_0)\ l\geq 1 \quad (4.1)$$

$$\frac{\mathrm{d}}{\mathrm{d}y_0}P_{l,2}(y_0)+\{((\Upsilon+1)\Omega)+\zeta_2(y_0)+\kappa\}P_{l,2}(y_0)=((\Upsilon+1)\Omega)\sum_{i=1}^{l} o_iP_{l-i,2}(y_0)\ l\geq 1 \quad (4.2)$$

$$\frac{\mathrm{d}}{\mathrm{d}y_0}H_{l,(}(y_0)+\{((\Upsilon+1)\Omega)+\eta(y_0)+\omega\}H_l(y_0)=((\Upsilon+1)\Omega)\sum_{i=1}^{\infty} o_iH_{l-i}(y_0)+\omega H_{l+1}l\geq 1 \quad (4.2)$$

$$\frac{\mathrm{d}}{\mathrm{d}y_0}H_0(y_0)+\big(((\Upsilon+1)\Omega)+\eta(y_0)\big)H_0(y_0)=\omega H_1(y_0) \quad (4.3)$$

$$\frac{\mathrm{d}}{\mathrm{d}y_0}M_l(y_0)+\{((\Upsilon+1)\Omega)+\phi(y_0)+\omega\}M_l(y_0)=((\Upsilon+1)\Omega)\sum_{i=1}^{l} o_iM_{l-i}(y_0)+\omega M_{l+1}l\geq 1 \quad (4.5)$$

$$\frac{\mathrm{d}}{\mathrm{d}y_0}M_0(y_0)+\big(((\Upsilon+1)\Omega)+\phi(y_0)\big)M_0(y_0)=\omega M_0(y_0) \quad (4.6)$$

$$\big((\Upsilon+1)\Omega\big)Q=(1-p)\int_0^\infty P_{0,2}(y_0)\zeta_2(y_0)\mathrm{d}y_0+\int_0^\infty M_0(y_0)\phi(y_0)\mathrm{d}y_0 + \int_0^\infty H_0(y_0)\eta(y_0)\mathrm{d}y_0 \quad (4.7)$$

As a result of solving these differential equations, the following results are satisfied:

$$P_{l,1}(0)=\big((\Upsilon+1)\Omega\big)o_lQ+(1-p)\int_0^\infty P_{l+1,2}(y_0)\zeta_2(y_0)\mathrm{d}y_0+\int_0^\infty H_{l+1,}(y_0)\eta(y_0)\mathrm{d}y_0$$

$$+\int_0^\infty M_l(y_0)\phi(y_0)dy_0, \qquad l\geq 1 \quad (4.8)$$

$$P_{l,2}(0)=\int_0^\infty P_{l,}(y_0)\zeta_1(y_0)\mathrm{d}y_0 \qquad l\geq 1 \quad (4.9)$$

$$M_l(0)=p\int_0^\infty P_{l+1,2}(y_0)\zeta_2(y_0)\mathrm{d}y_0, \qquad n\geq 0 \quad (4.10)$$

$$H_{l+1}(0)=\kappa\int_0^\infty P_{l,1}(y_0)\mathrm{d}y_0+\kappa\int_0^\infty P_{l,2}(y_0)\mathrm{d}y_0,=\kappa P_{l,1}+P_{l,2} \quad l\geq 0 \quad (4.11)$$

$$H_0(0)=0 \quad (4.12)$$

5. Random Period Queue Size Distribution

After summing over l from 1 to ∞ and multiplying Equations (4.1) - (4.6), by g^l, the result is

$$\frac{\mathrm{d}}{\mathrm{d}y_0}P_1(y_0,g)+\{(((\Upsilon+1)\Omega)-((\Upsilon+1)\Omega)O(g))+\zeta_1(y_0)+\kappa\}P_1(y_0,g)=0 \quad (5.1)$$

$$\frac{\mathrm{d}}{\mathrm{d}y_0}P_2(y_0,g)+\{(((\Upsilon+1)\Omega)-((\Upsilon+1)\Omega)O(g))+\zeta_2(y_0)+\kappa\}P_2(y_0,g)=0 \quad (5.2)$$

$$\frac{\mathrm{d}}{\mathrm{d}y}H(y_0,g)+\left\{((\Upsilon+1)\Omega)-((\Upsilon+1)\Omega)O(g)+\eta(y_0)+\omega-\frac{\omega}{g}\right\}H(y_0,g)=0 \quad (5.3)$$

$$\frac{\mathrm{d}}{\mathrm{d}y_0}M(y_0,g)+\left(((\Upsilon+1)\Omega)-((\Upsilon+1)\Omega)O(g)+\phi(y_0)+\omega-\frac{\omega}{g}\right)=0 \quad (5.4)$$

From the equations (5.1)-(5.4) were further integrated throughout the limits of 0 to y_0, obtaining the following results:

$$P_1(y_0,g)=P_1(0,g)\exp\left[(((\Upsilon+1)\Omega)-((\Upsilon+1)\Omega)O(g)+\kappa)y_0-\int_0^\infty\zeta_1(t)\mathrm{d}t\right] \quad (5.5)$$

$$P_2(y_0,g)=P_2(0,g)\exp\left[(((\Upsilon+1)\Omega)-((\Upsilon+1)\Omega)O(g)+\kappa)y_0-\int_0^\infty\zeta_2(t)\mathrm{d}t\right] \quad (5.6)$$

$$H(y_0,g)=H(0,g)\exp\left[\left(((\Upsilon+1)\Omega)-((\Upsilon+1)\Omega)O(g)+\omega-\frac{\omega}{g}\right)y_0-\int_0^\infty\eta(t)\mathrm{d}t\right] \quad (5.7)$$

$$M(y_0,g)=M(0,g)\exp\left[\left(((\Upsilon+1)\Omega)-((\Upsilon+1)\Omega)O(g)+\omega-\frac{\omega}{g}\right)y_0-\int_0^\infty\phi(t)\mathrm{d}t\right] \quad (5.8)$$

We multiply the boundary values of g^{l+1} and use the (PGFs).

$$gP_1(0,g)=(((\Upsilon+1)\Omega)O(g)-((\Upsilon+1)\Omega))Q+(1-p)\int_0^\infty P_2(y_0,g)\zeta_2(y_0)\mathrm{d}y_0$$

$$\int_0^\infty \mathrm{H}(y_0,g)\eta(y_0)\mathrm{d}y_0+\int_0^\infty \mathrm{M}(y_0,g)\phi(y_0)dy_0, \quad (5.9)$$

$$P_2(0,g)=\int_0^\infty P_1(y_0,g)\zeta_1(y_0)\mathrm{d}y_0 \quad (5.10)$$

$$g\mathrm{M}(0,g)=p\int_0^\infty P_2(y_0)\zeta_2(y_0)\mathrm{d}y_0, \quad (5.11)$$

$$H(0,g)=\kappa g[P_1(g)+P_2(g)] \quad (5.12)$$

We integrate the equations of (5.5) to y_0 provides us

$$P_1(g) = P_1(0,g)\left[\frac{1 - A_1(((\Upsilon+1)\Omega) - ((\Upsilon+1)\Omega)O(g) + \kappa)}{((\Upsilon+1)\Omega) - ((\Upsilon+1)\Omega)O(g) + \kappa}\right] \tag{5.13}$$

$$P_2(g) = P_2(0,g)\left[\frac{1 - A_2(((\Upsilon+1)\Omega) - ((\Upsilon+1)\Omega)O(g) + \kappa)}{((\Upsilon+1)\Omega) - ((\Upsilon+1)\Omega)O(g) + \kappa}\right] \tag{5.14}$$

$$M(g) = M(0,g)\left[\frac{1 - F\left(((\Upsilon+1)\Omega) - ((\Upsilon+1)\Omega)O(g) + \omega - \frac{\omega}{g}\right)}{((\Upsilon+1)\Omega) - ((\Upsilon+1)\Omega)O(g) + \omega - \frac{\omega}{g}}\right] \tag{5.15}$$

$$H(g) = H(0,g)\left[\frac{1 - B\left(((\Upsilon+1)\Omega) - ((\Upsilon+1)\Omega)O(g) + \omega - \frac{\omega}{g}\right)}{((\Upsilon+1)\Omega) - ((\Upsilon+1)\Omega)O(g) + \omega - \frac{\omega}{g}}\right] \tag{5.16}$$

where

$$A_1(((\Upsilon+1)\Omega) - ((\Upsilon+1)\Omega)O(g) + \kappa) = \int_0^\infty \mathrm{e}^{-(((\Upsilon+1)\Omega) - ((\Upsilon+1)\Omega)O(g) + \kappa)y_0}\,\mathrm{d}A_1(y_0)$$

$$A_2(((\Upsilon+1)\Omega) - ((\Upsilon+1)\Omega)O(g) + \kappa) = \int_0^\infty \mathrm{e}^{-(((\Upsilon+1)\Omega) - ((\Upsilon+1)\Omega)O(g) + \kappa)y_0}\,\mathrm{d}A_2(y_0)$$

$$B\left(((\Upsilon+1)\Omega) - ((\Upsilon+1)\Omega)O(g) + \omega - \frac{\omega}{g}\right) = \int_0^\infty \mathrm{e}^{-\left(((\Upsilon+1)\Omega) - ((\Upsilon+1)\Omega)O(g) + \omega - \frac{\omega}{g}\right)y_0}\,\mathrm{d}B(y_0)$$

$$F\left(((\Upsilon+1)\Omega) - ((\Upsilon+1)\Omega)O(g) + \omega - \frac{\omega}{g}\right) = \int_0^\infty \mathrm{e}^{-\left(((\Upsilon+1)\Omega) - ((\Upsilon+1)\Omega)O(g) + \omega - \frac{\omega}{g}\right)y_0}\,\mathrm{d}F(y_0)$$

We multiply the boundary values of g^{l+1}, and use the PGFs.

$$\int_0^\infty P_1(y_0,g)\zeta_1(y_0)\mathrm{d}y_0$$

$$\int_0^\infty P_2(y_0,g)\zeta_2(y_0)\mathrm{d}y_0$$

$$\int_0^\infty H(y_0,g)\eta(y_0)\mathrm{d}y_0$$

$$\int_0^\infty M(y_0,g)\phi(y_0)\mathrm{d}y_0$$

by combining the RHS of equations to (5.5 - 5.8) with $\zeta_1(y_0), \zeta_2(y_0), \eta(y_0)$, and $\phi(y_0)$ respectively, and integrating to y_0, we obtain

$$\int_0^\infty P_1(y_0,g)\zeta_1(y_0)\mathrm{d}y = P_1(0,g)A_1(((\Upsilon+1)\Omega)-((\Upsilon+1)\Omega)O(g)+\kappa) \quad (5.17)$$

$$\int_0^\infty P_2(y_0,g)\zeta_2(y_0)\mathrm{d}y = P_2(0,g)A_2(((\Upsilon+1)\Omega)-((\Upsilon+1)\Omega)O(g)+\kappa) \quad (5.18)$$

$$\int_0^\infty H(y_0,g)\eta(y_0)\mathrm{d}y = H(0,g)B\left(((\Upsilon+1)\Omega)-((\Upsilon+1)\Omega)O(g)+\omega-\frac{\omega}{g}\right) \quad (5.19)$$

$$\int_0^\infty M(y_0,g)\phi(y_0)\mathrm{d}y_0 = M(0,g)F\left(((\Upsilon+1)\Omega)-((\Upsilon+1)\Omega)O(g)+\omega-\frac{\omega}{g}\right) \quad (5.20)$$

Let us define,

$$((\Upsilon+1)\Omega)-((\Upsilon+1)\Omega)O(g)+\kappa = n;\ ((\Upsilon+1)\Omega)-((\Upsilon+1)\Omega)O(g)+\omega-\frac{\omega}{g} = k$$

Utilizing (5.17) - (5.20) in equations (5.9) - (5.11) we obtain

$$gP_1(0,g) = (((\Upsilon+1)\Omega)O(g)-((\Upsilon+1)\Omega))Q+(1-p)A_2(n)P_2(0,g) + H(0,g)B(k)+gM(0,g)F(k) \quad (5.21)$$

$$P_2(0,g) = P_1(0,g)A_1(n) \quad (5.22)$$

$$M(0,g) = pP_2(0,g)A_2(n) \quad (5.23)$$

Using (5.22) in (5.23) we get,

$$gM(0,g) = pP_1(0,g)A_1(n)A_2(n) \quad (5.24)$$

Again from (5.12) using (5.13) and (5.14) we get

$$H(0,g) = \frac{\kappa g}{n}\left[P_1(0,g)[1-A_1(n)]+P_2(0,g)[1-A_2(n)]\right] \quad (5.25)$$

Now using (5.22), (5.24), and (5.25) in Equation (5.21), we solve for $P_1(0,g)$

$$P_1(0,g) = \frac{n(((\Upsilon+1)\Omega)O(g)-((\Upsilon+1)\Omega))Q}{E(g)} \quad (5.26)$$

$$P_2(0,g) = \frac{n(((\Upsilon+1)\Omega)O(g)-((\Upsilon+1)\Omega))A_1(n)Q}{E(g)} \quad (5.27)$$

$$H(g) = \frac{\kappa g(((\Upsilon+1)\Omega)O(g)-((\Upsilon+1)\Omega))[1-A_1(n)A_2(n)]Q}{E(g)} \quad (5.28)$$

$$M(0,g) = \frac{pn(((\Upsilon+1)\Omega)O(g)-((\Upsilon+1)\Omega))A_1(n)A_2(n)Q}{E(g)} \quad (5.29)$$

Where,

$$E(g) = n[z-(1-p)A_1(n)A_2(n)-pA_1(n)A_2(n)F(k)]-\kappa gB(k)\{1-A_1(n)A_2(n)\}$$

Substituting the values from (5.26 - 5.29) in (5.13 - 5.16), we obtain

$$P_1(g) = \frac{(((\Upsilon+1)\Omega)O(g) - ((\Upsilon+1)\Omega))[1-A_1(n)]Q}{E(g)} \tag{5.30}$$

$$P_2(g) = \frac{(((\Upsilon+1)\Omega)O(g) - ((\Upsilon+1)\Omega))A_1(n)[1-A_2(n)]Q}{E(g)} \tag{5.31}$$

$$H(g) = \frac{\kappa g(((\Upsilon+1)\Omega)O(g) - ((\Upsilon+1)\Omega))[1-A_1(n)A_2(n)]Q}{E(g)} \cdot \left[\frac{1-B(k)}{k}\right] \tag{5.32}$$

$$M(g) = \frac{pn(((\Upsilon+1)\Omega)O(g) - ((\Upsilon+1)\Omega))A_1(n)A_2(n)Q}{E(g)} \cdot \left[\frac{1-F(k)}{k}\right] \tag{5.33}$$

The queue length is characterized by the probability-generating function $P_q(g)$, which remains independent of the system's behavior.

$$P_q(g) = P_1(g) + P_2(g) + H(g) + M(g) = \frac{V(g)}{E(g)} \tag{5.34}$$

By applying the normalization condition $P_q(1) + Q = 1$, we can calculate the probability of the idle period Q. Furthermore, we employ L'Hopital's Rule once more on equation (5.31) because it is indeterminate of the type $\frac{0}{0}$ at $g = 1$.

$$P_1(1) = \frac{((\Upsilon+1)\Omega)D(I)(1-A_1(\kappa))Q}{\begin{array}{c}-(((\Upsilon+1)\Omega)D(I) + \kappa(((\Upsilon+1)\Omega)D(I) - \omega)D(H)) \\ +[\kappa + ((\Upsilon+1)\Omega)D(I) + \kappa(((\Upsilon+1)\Omega)D(I) - \omega)D(H) \\ -p\kappa(((\Upsilon+1)\Omega)D(I) - \omega)D(M)]A_1(\kappa)A_2(\kappa)\end{array}} \tag{5.35}$$

The equation (5.35) denotes the probability of the server being busy in the first stage under a steady state behavior. The probability in a steady-state equation that the server is operating in the second stage is

$$P_2(1) = \frac{((\Upsilon+1)\Omega)D(I)A_1(\kappa)(1-A_2(\kappa))Q}{\begin{array}{c}-(((\Upsilon+1)\Omega)D(I) + \kappa(((\Upsilon+1)\Omega)D(I) - \omega)D(H)) \\ +[\kappa + ((\Upsilon+1)\Omega)D(I) + \kappa(((\Upsilon+1)\Omega)D(I) - \omega)D(H) \\ -p\kappa(((\Upsilon+1)\Omega)D(I) - \omega)D(M)]A_1(\kappa)A_2(\kappa)\end{array}} \tag{5.36}$$

where the expected of the arriving client group is $O(1) = 1, O'(1) = D(I)$, and $-B'(0) = D(H)$ represents the expected revamp period and the expected vacation period is $-F'(0) = D(M)$.

Thus, the normalization condition yields

$$Q = 1 - \frac{((\Upsilon+1)\Omega)D(I)[\{1+\kappa D(H)\} - \{1+\kappa D(H) - p\kappa D(M)\}A_1(\kappa)A_2(\kappa)]}{\kappa\omega D(H)\{1 - A_1(\kappa)A_2(\kappa)\} + p\kappa\omega A_1(\kappa)A_2(\kappa)} \tag{5.39}$$

Where,

$$\rho = \frac{((\Upsilon+1)\Omega)D(I)[\{1+\kappa D(H)\} - \{1+\kappa D(H) - p\kappa D(M)\}A_1(\kappa)A_2(\kappa)]}{\kappa\omega D(H)[1 - A_1(\kappa)A_2(\kappa)] + p\kappa\omega A_1(\kappa)A_2(\kappa)} < 1 \quad (5.40)$$

6. Performance Measures

Given that the expected queue length is of the indeterminate form $\frac{0}{0}$, we can apply L'Ho pital's Rule twice to calculate L_q, which is the derivative of the probability-generating function $P_q(g)$ for g evaluated at $g = 1$.

$$L_q = \lim_{g\to 1} \frac{E'(g)V''(g) - V'(g)E''(g)}{2(E'(g))^2} \quad (6.1)$$

At a random epoch, the expected queue length, denoted as L_q, can be determined. We can then use $W_q = \frac{L_q}{\lambda}$ to calculate the expected waiting period of the queue. Alternatively, we can compute the expected waiting period in the system using $W = \frac{L}{(\Upsilon+1)\Omega}$ and the expected queue length of the system using $L = L_q + \rho$.

6.1 Exponentially distributed vacation and service periods

In this scenario, we take the service period is exponentially distributed for the two service stages with service rates of $\zeta_1 > 0$ and $\zeta_2 > 0$, respectively. Additionally, the distribution of revamp period and vacation period are exponential, with revamp rates of $\eta > 0$ and vacation rates of $\phi > 0$, respectively.

Thus

$$A_1(n) = \frac{\zeta_1}{\zeta_1 + n}; A_2(n) = \frac{\zeta_2}{\zeta_2 + n}$$
$$B(k) = \frac{\eta}{\eta + k}; F(k) = \frac{\phi}{\phi + k}$$
$$D(H) = \frac{1}{\eta}; D(H^2) = \frac{2}{\eta^2}$$
$$D(M) = \frac{1}{\phi}; D(M^2) = \frac{2}{\phi^2}$$

where

$$n = \big((\Upsilon+1)\Omega\big) - \big((\Upsilon+1)\Omega\big)O(g) + \kappa; k = \big((\Upsilon+1)\Omega\big) - \big((\Upsilon+1)\Omega\big)O(g) + \omega - \frac{\omega}{g} \text{ where}$$

$$E(g) = (((\Upsilon+1)\Omega) - ((\Upsilon+1)\Omega)O(g) + \kappa)\left[g - \left\{(1-p) + p\frac{\phi}{\phi+k}\right\}\frac{\zeta_1\zeta_2}{(\zeta_1+n)(\zeta_2+n)}\right]$$
$$- \kappa g\left\{1 - \frac{\zeta_1\zeta_2}{(\zeta_1+n)(\zeta_2+n)}\right\}\frac{\eta}{\eta+k}$$

Utilizing the above relations in equations (5.30) - (5.33), we obtain

$$P_1(g) = \frac{[((\Upsilon+1)\Omega)O(g) - ((\Upsilon+1)\Omega)]\left[1-\frac{\zeta_1}{\zeta_1+n}\right]Q}{E(g)} \tag{6.2}$$

$$P_2(g) = \frac{[((\Upsilon+1)\Omega)O(g) - ((\Upsilon+1)\Omega)]\frac{\zeta_1}{\zeta_1+n}\left[1-\frac{\zeta_2}{\zeta_2+n}\right]Q}{E(g)} \tag{6.3}$$

$$H(g) = \frac{\kappa[((\Upsilon+1)\Omega)O(g) - ((\Upsilon+1)\Omega)]\left[1-\frac{\zeta_1\zeta_2}{(\zeta_1+n)(\zeta_2+n)}\right]\frac{1}{(\eta+k)}Q}{E(g)} \tag{6.4}$$

$$M(g) = \frac{pn[((\Upsilon+1)\Omega)O(g) - ((\Upsilon+1)\Omega)]\frac{\zeta_1\zeta_2}{(\zeta_1+n)(\zeta_2+n)(\phi+k)}Q}{E(g)} \tag{6.5}$$

Consequently, the probability that the server will be offering first-stage services at an arbitrary period is

$$P_1(1) = \frac{((\Upsilon+1)\Omega)D(I)\left[1-\frac{\zeta_1}{\zeta_1+\kappa}\right]Q}{\begin{array}{c}-\left[((\Upsilon+1)\Omega)D(I)+\frac{\kappa(((\Upsilon+1)\Omega)D(I)-\omega)}{\eta}\right] \\ +[\kappa+((\Upsilon+1)\Omega)E(I) \\ +\kappa(((\Upsilon+1)\Omega)D(I)-\omega)\left\{\frac{1}{\eta}-\frac{p}{\phi}\right\}\Big]\frac{\zeta_1\zeta_2}{(\zeta_1+\kappa)(\zeta_2+\kappa)}\end{array}} \tag{6.6}$$

Consequently, the probability that the server will be offering second-stage services at an arbitrary
period is

$$P_2(1) = \frac{((\Upsilon+1)\Omega)E(I)\frac{\zeta_1}{(\zeta_1+\kappa)}\left\{1-\frac{\zeta_2}{\zeta_2+\kappa}\right\}Q}{\begin{array}{c}-\left[((\Upsilon+1)\Omega)D(I)+\frac{\kappa(((\Upsilon+1)\Omega)D(I)-\omega)}{\eta}\right] \\ +[\kappa+((\Upsilon+1)\Omega)E(I) \\ +\kappa(((\Upsilon+1)\Omega)D(I)-\omega)\left\{\frac{1}{\eta}-\frac{p}{\phi}\right\}\Big]\frac{\zeta_1\zeta_2}{(\zeta_1+\kappa)(\zeta_2+\kappa)}\end{array}} \tag{6.7}$$

The server may need maintenance at some random moment.

$$H(1)=\frac{\left[\frac{\kappa((\Upsilon+1)\Omega)D(I)}{\eta}\right]\left[1-\frac{\zeta_1\zeta_2}{(\zeta_1+\kappa)(\zeta_2+\kappa)}\right]Q}{\begin{gathered}-\left[((\Upsilon+1)\Omega)D(I)+\frac{\kappa(((\Upsilon+1)\Omega)D(I)-\omega)}{\eta}\right]\\+[\kappa+((\Upsilon+1)\Omega)D(I)\\+\kappa(((\Upsilon+1)\Omega)D(I)-\omega)\left\{\frac{1}{\eta}-\frac{p}{\phi}\right\}\Big]\frac{\zeta_1\zeta_2}{(\zeta_1+\kappa)(\zeta_2+\kappa)}\end{gathered}}\tag{6.8}$$

The probability of the server being under vacation at any random epoch in the period is expressed as:

$$M(1)=\frac{p\kappa\left[\frac{((\Upsilon+1)\Omega)D(I)}{\phi}\right]\frac{\zeta_1\zeta_2}{(\zeta_1\kappa)(\zeta_2+\kappa)}Q}{\begin{gathered}-\left[((\Upsilon+1)\Omega)D(I)+\frac{\kappa(((\Upsilon+1)\Omega)D(I)-\omega)}{\eta}\right]\\+[\kappa+((\Upsilon+1)\Omega)D(I)\\+\kappa(((\Upsilon+1)\Omega)D(I)-\omega)\left\{\frac{1}{\eta}-\frac{p}{\phi}\right\}\Big]\frac{\zeta_1\zeta_2}{(\zeta_1+\kappa)(\zeta_2+\kappa)}\end{gathered}}\tag{6.9}$$

The probability that the server is free but available within the system is expressed as:

$$Q=1-\big((\Upsilon+1)\Omega\big)D(I)\frac{\left[\left\{1+\frac{\kappa}{\eta}\right\}-\left\{1+\frac{\kappa}{\eta}-\frac{p\kappa}{\phi}\right\}\frac{\zeta_1\zeta_2}{(\zeta_1+\kappa)(\zeta_2+\kappa)}\right]}{\frac{\kappa\omega}{\eta}\left\{1-\frac{\zeta_1\zeta_2}{(\zeta_1+\kappa)(\zeta_2+\kappa)}\right\}+p\kappa\omega\frac{\zeta_1\zeta_2}{(\zeta_1+\kappa)(\zeta_2+\kappa)}}\tag{6.10}$$

Similar to this, $V'(1), V''(1), E'(1)$, and $E''(1)$ may be identified and utilized in equation (6.1). This will provide the expected queue length and expected waiting period.

$$V'(1)=Q\left[((\Upsilon+1)\Omega)D(I)\left\{1+\frac{\kappa}{\eta}\right\}-\lambda D(I)\left\{1+\frac{\kappa}{\eta}-\frac{p\kappa}{\phi}\right\}\frac{\zeta_1\zeta_2}{(\zeta_1+\kappa)(\zeta_2+\kappa)}\right]$$

$$\begin{aligned}V''(1)=&\left[\left[((\Upsilon+1)\Omega)D(I/(I-1))\left\{\left(1+\frac{\kappa}{\eta}\right)\left(1-\frac{\zeta_1\zeta_2}{(\zeta_1+\kappa)(\zeta_2+\kappa)}\right)+\frac{p\kappa}{\phi}\frac{\zeta_1\zeta_2}{(\zeta_1+\kappa)(\zeta_2+\kappa)}\right\}\right]\right]\\&-2((\Upsilon+1)\Omega)D(I)\left\{\frac{\kappa(((\Upsilon+1)\Omega)D(I)-\omega)}{\eta^2}\left(1-\frac{\zeta_1\zeta_2}{(\zeta_1+\kappa)(\zeta_2+\kappa)}\right)\right\}\\&+\left\{\frac{p(((\Upsilon+1)\Omega)D(I)-\omega)}{\phi^2}\frac{\zeta_1\zeta_2}{(\zeta_1+\kappa)(\zeta_2+\kappa)}\right\}\end{aligned}$$

$$\begin{aligned}E'(1)=&-\left[((\Upsilon+1)\Omega)D(I)+\frac{\kappa(\lambda D(I)-\omega)}{\eta}\right]\\&+\left\{\kappa+((\Upsilon+1)\Omega)D(I)+\frac{\kappa(((\Upsilon+1)\Omega)D(I)-\omega)}{\eta}-\frac{p\kappa(((\Upsilon+1)\Omega)D(I)-\omega)}{\phi}\right\}\frac{\zeta_1\zeta_2}{(\zeta_1+\kappa)(\zeta_2+\kappa)}\end{aligned}$$

$$E''(1)=-((\Upsilon+1)\Omega)D(I/(I-1))\left\{\left(1+\frac{\kappa}{\eta}\right)+\frac{\zeta_1\zeta_2}{(\zeta_1+\kappa)(\zeta_2+\kappa)}\left(1-\frac{\kappa}{\eta}+\frac{p\kappa}{\phi}\right)\right\}$$

$$-2\left[\frac{\kappa\omega}{\eta}+\frac{(((\Upsilon+1)\Omega)D(I)-\omega)^2}{\eta^2}\right]\left[1-\frac{\zeta_1\zeta_2}{(\zeta_1+\kappa)(\zeta_2+\kappa)}\right]-2p\kappa\left[\frac{\omega}{\phi}+\frac{(((\Upsilon+1)\Omega)D(I)-\omega)^2}{\phi^2}\right]\frac{\zeta_1\zeta_2}{(\zeta_1+\kappa)(\zeta_2+\kappa)}-\left\{((\Upsilon+1)\Omega)D(I)+\frac{\kappa(((\Upsilon+1)\Omega)D(I)-\omega)}{\eta}-\frac{p\kappa(((\Upsilon+1)\Omega)D(I)-\omega)}{\phi}+\kappa\right\}\left\{\frac{\zeta_1\zeta_2}{(\zeta_1+\kappa)^2(\zeta_2+\kappa)}+\frac{\zeta_1\zeta_2}{(\zeta_1+\kappa)(\zeta_2+\kappa)^2}\right\}$$

7. Numerical Illustration

Numerical data are generated to observe the impact of various factors, particularly parameters related to reneging and failures, on server states such as vacation duration, utilization factor, and probability of idle server period. We consider mathematically possible solutions by assuming that the exponential distribution follows the service period, revamp period, and vacation period. The values of the parameters of this queuing model are given. To assess the effects of reneging

Ω	ζ	$((\Upsilon+1)\Omega)$	ζ_1	ζ_2	η	ϕ	p	ω	κ
2	0.1	2.2	4	8	10	7	0.5	5,8,10	1,2,3,4

(ω) and failure (κ) on the queuing model, we present the results in Table 1.

ω	κ	Q	ρ	L_q	L	W_q	W
5	1	0.5782	0.4218	4.3258	4.7476	1.9663	2.1580
	2	0.5533	0.4467	3.6746	4.1212	1.6703	1.8733
	3	0.5324	0.4676	3.4534	3.9210	1.5697	1.7823
	4	0.5151	0.4849	3.2702	3.7551	1.4864	1.7068
8	1	0.7364	0.2636	2.7143	2.9779	1.2338	1.3536
	2	0.7208	0.2792	2.0759	2.3551	0.9436	1.0705
	3	0.7078	0.2922	1.7564	2.0486	0.7984	0.9312
	4	0.6969	0.3031	1.5385	1.8415	0.6993	0.8371
10	1	0.7891	0.2109	2.2956	2.5065	1.0435	1.1393
	2	0.7767	0.2233	1.7035	1.9269	0.7743	0.8758

	3	0.7662	0.2338	1.3959	1.6297	0.6345	0.7408
	4	0.7576	0.2424	1.1936	1.4361	0.5426	0.6528

Table 1: The table illustrates that the parameter representing client impatience ω increases, the utilization factor also increases for various values of failure κ. Simultaneously, the server's idle period probability decreases, resulting in a decrease in both the expected queue length L_q and the expected waiting period W_q.

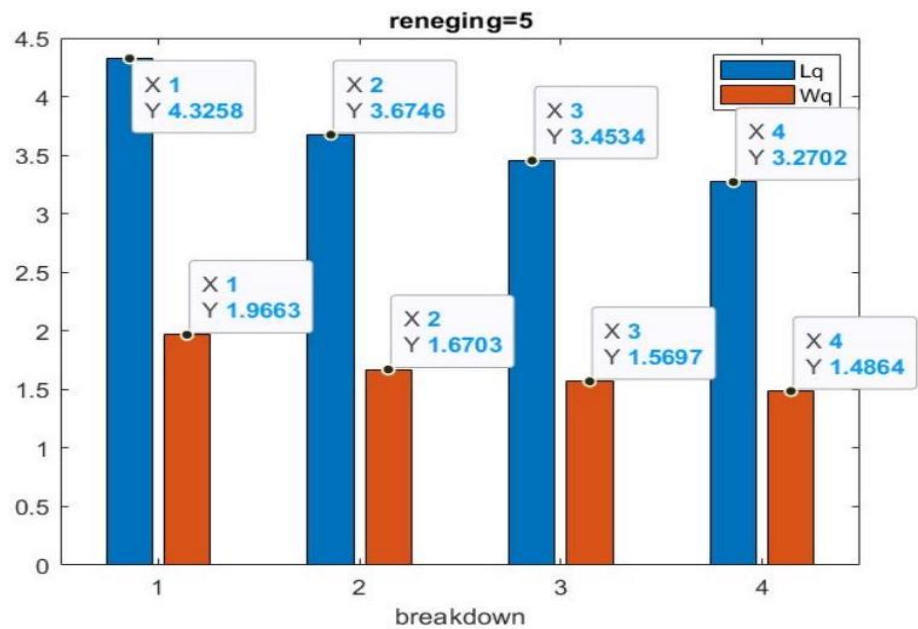

Figure 1: The above figure shows that the effect of reneging takes 5 and failure occurs in the ranges from 1 to 4 on the expected queue length L_q, expected waiting period W_q decreases.

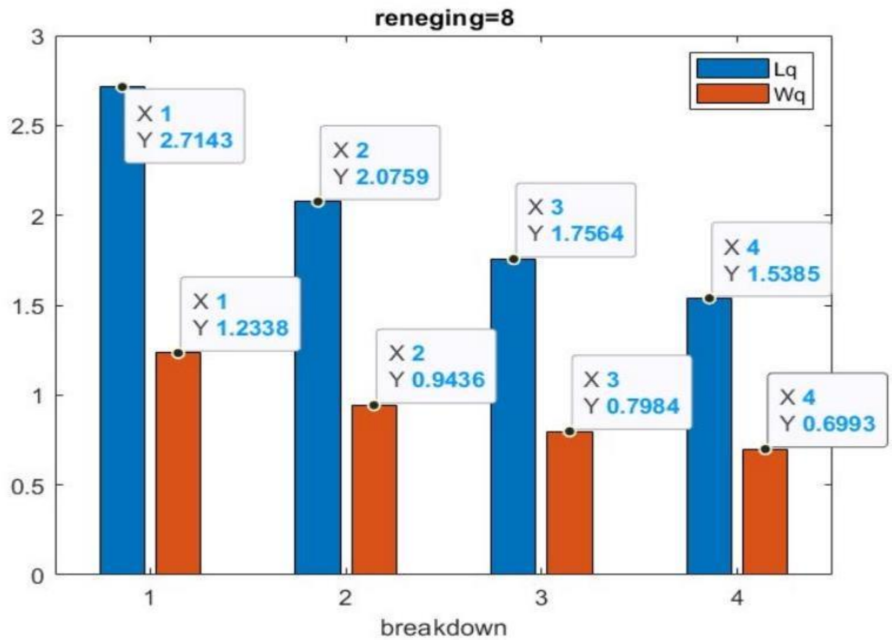

Figure 2: The figure illustrates how the expected queue length L_q and expected waiting period W_q decreases as the effect of reneging takes 8 , with failure occurring the ranges from 1 to 4 .

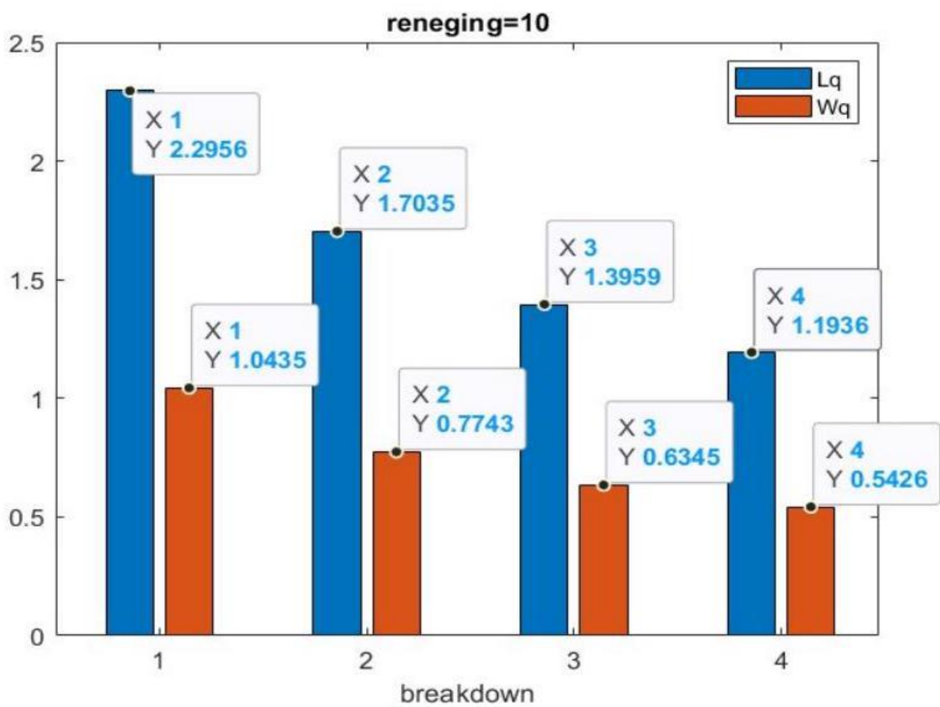

Figure 3: The figure shows that the effect of reneging takes 10 and failure occurs in the ranges from 1 to 4 on the expected queue length L_q, expected waiting period W_q decreases.

8. Comparison Table

In this section, the encouraged queue length distribution, expected queue length L_q, and expected waiting period W_q demonstrate greater effectiveness compared to Poisson arrival, especially with a 10% discount.

ω	κ	Poisson arrival		Encouraged arrival	
		of L_q	of W_q	of L_q	of W_q
5	1	2.8350	1.4175	4.3258	1.9663
	2	2.4236	1.2118	3.6746	1.6703
	3	2.3144	1.1572	3.4534	1.5697
	4	1.1716	0.5858	3.2702	1.4864
8	1	1.9723	0.9861	2.7143	1.2338
	2	1.5610	0.7805	2.0759	0.9436
	3	1.3509	0.7443	1.7564	0.7984
	4	1.2041	0.6021	1.5385	0.6993
10	1	2.0494	1.0247	4.3258	1.0435
	2	1.3329	0.6665	3.6746	0.7743
	3	1.2210	0.6105	3.4534	0.6345
	4	1.2170	0.6085	3.2702	0.5426

Table 2: The following table shows the comparison of Poisson and encouraged arrival for 10% discount.

9. Cost Analysis

To optimize the system's operating cost, we established a cost analysis approach. The system with encouraged arrivals is characterized by $\Omega = 2$, where $\Upsilon = 0.1$ denotes the expected number of clients arriving for a specific service within a defined period. If the payment is processed too slowly, the client will leave the queue. In this situation, the firm will be deprived of the amount 's' per cost for the suitable period duration. Let's assume that one server will cost Rs. 25 to operate. As part of our analysis, of the queuing system, we evaluated the operating cost (OP) and waiting cost for the encouraged arrival of 10% discount.

ω	κ	Encouraged arrival Rate	Waiting period	Waiting cost	Operating cost	Total cost
5	1	2.2	1.9663	3.9326	25	28.9326
	2	2.2	1.6703	3.3406	25	28.3406
	3	2.2	1.5697	3.1394	25	28.1394
	4	2.2	1.4864	2.9728	25	27.9728

Table 3: The following table shows the total cost of encouraged arrival for 10% discount

ω	κ	Encouraged arrival Rate	Waiting period	Waiting cost	Operating cost	Total cost
8	1	2.2	1.2338	2.4676	25	27.4676
	2	2.2	0.9436	1.8872	25	26.8872
	3	2.2	0.7984	1.5968	25	26.5968
	4	2.2	0.6993	1.3986	25	26.3986

Table 4: The following table shows the total cost of encouraged arrival for 10% discount

ω	κ	Encouraged arrival Rate	Waiting period	Waiting cost	Operating cost	Total cost
10	1	2.2	1.0435	2.087	25	27.087
	2	2.2	0.7743	1.5486	25	26.5486
	3	2.2	0.6345	1.269	25	26.269
	4	2.2	0.5426	1.0852	25	26.0852

Table 5: The following table shows the total cost of encouraged arrival for 10% discount

10. Results and Discussion

- In Table 1, it is observed that increasing reneging (ω), for different values of the failure (κ) leads to an increase in the utilization factor, while the probability of a server-free period decreases.
- Table 2 compares the Poisson and the encouraged arrival systems for a 10% discount. It shows that, compared to the Poisson arrival system, the encouraged arrival queuing system experiences increased queue lengths.
- Figures 1, 2, and 3 depict the impact of failure and failure on the expected queue length $\left(L_q\right)$ and expected waiting period $\left(W_q\right)$. It is evident that as failure occurs, it decreases the expected queue length, the same as the expected waiting period decreases because of reneging from the queue.
- Table 5 indicates the optimal performance that is achieved when the values of ω and κ are minimized to 10 and 4, respectively. This total cost analysis proves to be more efficient for optimizing cost analysis.

11. Conclusion

We investigated a two-phase heterogeneous service facility queue with failures, server vacations, and a recommended encouraged queue length distribution in this study. By employing the supplementary variable approach, we obtained steady-state results and analyzed the expected encouraged queue length and expected client waiting period. We enhanced the queue size by introducing encouraged arrivals based on queue length distribution. The table 5 demonstrates that the optimal performance is achieved when the values of ω and κ are minimized to 10 and 4 , respectively. Total cost analysis proves to be efficient for optimizing cost analysis.

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